

HILBERT-MUMFORD CRITERION FOR NODAL CURVES

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ABSTRACT. We prove by Hilbert-Mumford criterion that a slope stable polarized weighted pointed nodal curve is Chow asymptotic stable. This generalizes the result of Caporaso on stability of polarized nodal curves, and of Hassett on weighted pointed stable curves polarized by the weighted dualizing sheaves. It also solved a question raised by Mumford and Gieseker to prove the Chow asymptotic stability of stable nodal curves by Hilbert-Mumford criterion.

1. INTRODUCTION AND SUMMARY OF MAIN RESULT

In late seventy, Mumford [16] and Gieseker [7] constructed the coarse moduli space $\overline{\mathcal{M}}_g$ of stable curves using Mumford's Geometric Invariant Theory (GIT). They proved the stability of smooth curves by verifying Hilbert-Mumford stability criterion; for nodal curves, they proved the stability indirectly by using semi-stable replacement and the numerical criterion to rule out curves with worse than nodal singularities. This construction has been very successful and is widely adopted subsequently for studying related to stability of curves, for instance, Caporaso's proof of asymptotic stability of nodal curves [2].

In this paper, we will prove the Chow asymptotic stability of weighted pointed nodal curves by verifying Hilbert-Mumford criterion directly. As an application, we provide a GIT construction of the moduli of weighted pointed stable curves. An interesting consequence of this construction is that the GIT closure of the moduli of weighted pointed smooth curves, using Chow asymptotic stability, is identical to Hassett's coarse moduli of weighted pointed stable curves; nevertheless, its universal family includes strictly semistable weighted pointed nodal curves.

Another application of our stability study is showing that a polarized nodal curve is K -stable (c.f. Section 7) if and only if the polarization is numerically equivalent to a multiple of its dualizing sheaf. This generalizes a theorem of Odaka that a stable nodal curve polarized with dualizing sheaf is K -stable.

The primary goal of this work is to understand the GIT compactification of moduli of canonically polarized varieties. The recent work on the relation between various notions of K -stabilities and the existence of constant scalar metrics suggests that some deep and interesting geometry are yet to be uncovered in this area. This work is a first step toward this direction. We hope this study will help us understand the stability of high dimensional singular varieties.

We now outline the results proved in this paper.

Definition 1.1 (Hassett [9]). *A weighted pointed nodal curve $(X, \mathbf{x}, \mathbf{a})$ consists of a reduced, connected curve X , n ordered (not necessarily distinct) smooth points $\mathbf{x} = (x_1, \dots, x_n)$ of X , and weights $\mathbf{a} = (a_1, \dots, a_n)$, $a_i \in \mathbb{Q}_{\geq 0}$, of \mathbf{x} , such that the total weight at any point*

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is no more than one (i.e. for any $p \in X$, $\sum_{x_i=p} a_i \leq 1$). A polarized weighted pointed curve is a weighted pointed curve together with a polarization $\mathcal{O}_X(1)$.

In this paper, we will use $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ to denote such a polarized weighted pointed curve. In case $\mathcal{O}_X(1)$ is very ample, we let

$$(1.1) \quad \iota : X \xrightarrow{\subset} \mathbb{P}W, \quad W = H^0(\mathcal{O}_X(1))^\vee,$$

be the tautological embedding; let

$$\text{Chow}(X) \in \text{Div}^{d,d}[(\mathbb{P}W^\vee)^2] := \mathbb{P}H^0(\mathcal{O}_{(\mathbb{P}W^\vee)^2}(d, d))$$

be the Chow point of X , which is the bi-degree (d, d) hypersurface in $(\mathbb{P}W^\vee)^2$ consisting of points $(V_1, V_2) \in (\mathbb{P}W^\vee)^2$ such that $V_i \subset \mathbb{P}W$ are hyperplanes satisfying $V_1 \cap V_2 \cap \iota(X) \neq \emptyset$. We abbreviate

$$(1.2) \quad \Xi := \text{Div}^{d,d}[(\mathbb{P}W^\vee)^2] \times (\mathbb{P}W)^n,$$

and let the Chow point of $(X, \mathcal{O}_X(1), \mathbf{x})$ be

$$\text{Chow}(X, \mathbf{x}) = (\text{Chow}(X), \mathbf{x}) \in \Xi.$$

The stability of this Chow point is tested by the positivity of the \mathbf{a} -weight of any one parameter subgroup $\lambda : \mathbb{C}^\times \rightarrow SL(W)$. (A one parameter subgroup, abbreviated to 1-PS, is always non-trivial.) Given a 1-PS λ , its action on W induces an action on Ξ . Since $\text{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is a projective space, it has a canonical polarization $\mathcal{O}(1)$. We let

$$\mathcal{O}_\Xi(1, \mathbf{a})$$

be the \mathbb{Q} -ample line bundle on Ξ that has degree 1 on $\text{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ and has degree a_i on the i -th copy of the $\mathbb{P}W$ in $(\mathbb{P}W)^n$. Integral multiple of this line bundle is canonically linearized by $SL(W)$.

Definition 1.2. With $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ understood, we define the \mathbf{a} - λ -weight of $\text{Chow}(X, \mathbf{x}) \in \Xi$ be the weight of the λ -action on the fiber $\mathcal{O}_\Xi(1, \mathbf{a})|_\zeta$, where $\zeta = \lim_{t \rightarrow 0} \lambda(t) \cdot \text{Chow}(X, \mathbf{x}) \in \Xi$; we denote this weight to be $\omega_{\mathbf{a}}(\lambda)$.

We define $\omega(\lambda)$ be the λ -weight of $\text{Chow}(X) \in \text{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ defined with $\text{Chow}(X, \mathbf{x})$ (resp. $\mathcal{O}_\Xi(1, \mathbf{a})$) replaced by $\text{Chow}(X)$ (resp. $\mathcal{O}(1)$).

Definition 1.3. Given $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$, we say that it is stable (resp. semistable) if for any 1-PS λ of $SL(W)$, the \mathbf{a} - λ -weight of $\text{Chow}(X, \mathbf{x})$ is positive (resp. non-negative).

To make an analogy with the slope stability of vector bundle, we introduce the notion of slope stable by testing on proper closed subcurves $Y \subset X$. First, with $\mathcal{O}_X(1)$ understood, we denote $\deg X = \deg \mathcal{O}_X(1)$, and for any subcurve $Y \subset X$, we denote $\deg Y = \deg \mathcal{O}_X(1)|_Y$. For any proper subcurve $Y \subset X$, we define the number of linking nodes of Y to be

$$(1.3) \quad \ell_Y = |Y \cap Y^{\mathfrak{c}}|, \quad Y^{\mathfrak{c}} = \overline{X \setminus Y}.$$

Definition 1.4. Given $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$, We say that it is slope (semi-)stable if X is nodal and if for any proper subcurve $Y \subsetneq X$ we have

$$(1.4) \quad \frac{\deg Y + \frac{\ell_Y}{2} + \sum_{x_j \in Y} \frac{a_j}{2}}{h^0(\mathcal{O}_X(1)|_Y)} < \frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{h^0(\mathcal{O}_X(1))}, \quad (\text{resp. } \leq).$$

In this paper, we will prove by verifying the Hilbert-Mumford criterion the following theorem. For the weight \mathbf{a} and $g(X) = g$, we denote

$$(1.5) \quad \chi_{\mathbf{a}}(X) := g - 1 + (a_1 + \cdots + a_n).$$

Theorem 1.5. *Given g and \mathbf{a} such that $\chi_{\mathbf{a}}(X) > 0$, there is an N so that a genus g polarized weighed pointed curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ such that $\deg X \geq N$ is (semi-)stable if and only if it is slope (semi-)stable.*

For (X, \mathbf{x}) , we abbreviate the \mathbb{Q} -line bundle $\omega_X(\sum a_i x_i)$ to $\omega_X(\mathbf{a} \cdot \mathbf{x})$. For integer k so that $k \cdot a_i \in \mathbb{Z}$ for all i , then $\omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k} = \omega_X^{\otimes k}(\sum k a_i x_i)$ is a line bundle. In Section 5, we will show that in case $\deg X$ is sufficiently large, the slope stability is equivalent to the criterion:

Proposition 1.6. *Given g and \mathbf{a} such that $\chi_{\mathbf{a}}(X) > 0$, there is an N so that a genus g polarized weighed pointed curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ such that $\deg X \geq N$ is slope (semi-)stable if and only if for any proper subcurve $Y \subsetneq X$ satisfying $h^0(\mathcal{O}_X(1)|_Y) < h^0(\mathcal{O}_X(1))$, we have*

$$(1.6) \quad \left| \left(\deg Y + \sum_{x_j \in Y} \frac{a_j}{2} \right) - \frac{\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x})}{\deg \omega_X(\mathbf{a} \cdot \mathbf{x})} \left(\deg X + \sum_{j=1}^n \frac{a_j}{2} \right) \right| < \frac{\ell_Y}{2}, \text{ (resp. } \leq \text{)}.$$

The case $\mathbf{x} = \emptyset$ is a theorem of Caporaso [2] on the stability of polarized nodal curves. The case of the asymptotic Hilbert stability of smooth¹ weighted pointed curves is a theorem of David Swinarski [22] (see also [15]).

We now sketch the main ingredients of our proof. Our starting point is a theorem of Mumford that expresses the \mathbf{a} - λ -weight of Chow (X, \mathbf{x}) in terms of the leading coefficient of the Hilbert-Samuel polynomial of an ideal $\mathcal{J} \subset \mathcal{O}_{X \times \mathbb{A}^1}(1)$ (cf. Prop. 2.1). Our first observation is that this leading coefficient can be evaluated by the leading coefficient of the Hilbert-Samuel polynomial of the pull back $\tilde{\mathcal{J}}$ of \mathcal{J} to the normalization \tilde{X} of X . This transforms the evaluation of the \mathbf{a} - λ -weight to the calculation of the areas of a class of Newton polygons associated to the pull back sheaf $\tilde{\mathcal{J}}$. By dividing the Newton polygons into two kinds and studying them separately, we obtain an effective bound of the areas, thus a bound of the \mathbf{a} - λ -weight of Chow (X, \mathbf{x}) . This bound is linear in the weights of λ . We then apply linear programming to complete our proof of Theorem 1.5.

Our GIT construction of the moduli of weighted pointed stable curves goes as follows. We form the Hilbert scheme \mathcal{H} of pointed 1-dimensional subscheme of \mathbb{P}^m of fixed degree. Let $\psi : \mathcal{H} \rightarrow \mathcal{C}$ be the Hilbert-Chow morphism (map) to the Chow variety of pointed 1-dimensional cycles in \mathbb{P}^m of the same degree, equivariant under $SL(m+1)$. Applying our main theorem, we conclude that in case the degree is sufficiently large, the preimage under ψ of the set $\mathcal{C}^{ss} \subset \mathcal{C}$ of GIT-semistable points is the set of semistable polarized weighted pointed nodal curves. Let $\mathcal{K} \subset \mathcal{H}$ be the subset of canonically polarized weighted pointed smooth curves. We prove that the GIT-quotient of the closure $\overline{\mathcal{K}}$ is isomorphic to the Hassett's moduli of weighted pointed stable curves. An interesting observation is that the complement $\overline{\mathcal{K}} - \mathcal{K}$ contains polarized semistable but not canonically polarized weighted pointed curves. Thus though GIT gives the same compactification as that of Hassett of the moduli of canonically polarized weighted pointed smooth curves, the geometric objects added to obtain the compactification in the mentioned two constructions are markedly different. It is worth pursuing to see how this extends in the high dimensional case.

In the end, using that the Donaldson-Futaki invariants can be expressed as the limit of certain Chow weights under a 1-PS, we apply our main theorem to prove that a polarized nodal curve $(X, \mathcal{O}_X(1))$ is K -stable if and only if $\mathcal{O}_X(1)$ is numerically equivalent to a multiple of ω_X . This implies that GIT compactification is same as the compactification of

¹Notice if X is smooth then (1.6) becomes vacuous.

smooth curve using K -stability. This is analogous to that the Uhlenbeck compactification coincides with the GIT compactification of the moduli of vector bundles over curves.

The paper is organized as follows. In Section two, we show that the weights can be evaluated via the leading coefficients of the Hilbert-Samuel polynomial of a sheaf on the normalization \tilde{X} . In Section three, we reduce our study to a particular class of 1-PS: the staircase 1-PS. We will derive a sharp bound for each irreducible component in Section four. We complete the proof of our main theorems in Section five. The last two sections include the application of our stability study to constructing moduli of weighted pointed curves and to study the K -stability of polarized curves.

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List of notations

$\mathcal{J}(\lambda); \tilde{\mathcal{J}}(\lambda)$	$(t^{\rho_0} s_0, \dots) \subset \mathcal{O}_{X \times \mathbb{A}^1}(1)$; similarly defined on \tilde{X}	(2.3)
$e(\mathcal{J}(\lambda)); e(\tilde{\mathcal{J}})$	n.l.c. $\chi(\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}(\lambda)^k)$; similarly defined on \tilde{X}	(2.4); after (2.6)
$e(\tilde{\mathcal{J}})_q; e(\tilde{\mathcal{J}}_\alpha)$	contribution of $e(\tilde{\mathcal{J}})$ at $q \in X$; along \tilde{X}_α	(2.12)
$\omega(\lambda)$	λ -Chow weight	Prop. 2.1
$v(\tilde{s}_i, q)$	the vanishing order of \tilde{s}_i at q	(2.8)
$\tilde{h}(q)$	$\max\{i \mid v(\tilde{s}_i, q) \neq \infty\}$	(2.9)
\tilde{h}_α	$\min_i\{i \mid \tilde{s}_j _{\tilde{X}_\alpha} = 0, \text{ for } j \geq i+1\}$.	(2.15)
Δ_q	Newton polygon supported at $q \in \tilde{X}$	Def. 2.6
$\mathcal{E}_i = \mathcal{E}(\lambda)_i$	$(s_i, s_{i+1}, \dots, s_m) \subset \mathcal{O}_X(1)$	(3.1)
$\Lambda_\alpha(\lambda); \Lambda(\lambda)$	$\{q \in X_\alpha \mid s_{\tilde{h}_\alpha}(q) = 0\}$; $\Lambda(\lambda) = \cup_{\alpha=1}^r \Lambda_\alpha(\lambda)$	Def. 3.1
$\delta(\tilde{s}_i, p)$	$\text{length}(\tilde{\mathcal{E}}_i/\tilde{\mathcal{E}}_{i+1})_p$ or $= 0$	Def. 3.2
$\text{inc}_\alpha(\tilde{s}_i)$	$\sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p)p$ and $\text{inc}(\tilde{s}_i) = \sum_\alpha \text{inc}_\alpha(\tilde{s}_i)$	Def. 3.1
$\delta_\alpha(\tilde{s}_i); \delta(\tilde{s}_i)$	$\sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p); \delta(\tilde{s}_i) = \sum_\alpha \delta_\alpha(\tilde{s}_i)$	Def. 3.1
$w(\tilde{\mathcal{E}}_i, p); w_\alpha(\tilde{\mathcal{E}}_i)$	$\text{length}(\mathcal{O}_{\tilde{X}}(1)/\tilde{\mathcal{E}}_i)_p$; $w_\alpha(\tilde{\mathcal{E}}_i) = \sum_{p \in \tilde{X}_\alpha} w(\tilde{\mathcal{E}}_i, p)$	Def. 3.2
$\mathbb{I}_\alpha = \mathbb{I}_\alpha(\lambda)$	$\{i \in \mathbb{I} \mid \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \neq \emptyset \text{ or } i = \tilde{h}_\alpha\}$	(3.3)
$L_Y; L_\alpha; \tilde{L}_Y; \tilde{L}_\alpha$	$Y \cap Y^\mathbb{C}; L_{X_\alpha}; \pi^{-1}(L_Y) \cap \tilde{Y}; \tilde{L}_{X_\alpha}$	(3.9) and (3.11)
$\tilde{N}_Y; N_\alpha; \tilde{N}_\alpha$	$\pi^{-1}(N_Y) \cap \tilde{Y}; N_{X_\alpha}; \tilde{N}_{X_\alpha}$	(3.10)
$\ell_\alpha; \ell_{\alpha, \beta}; \ell_{\alpha, \alpha}$	$ L_\alpha ; X_\alpha \cap X_\beta ; - L_\alpha $	(3.11); (6.9)
$\mathbb{I}_\alpha^{\text{pri}}$	$\{i \in \mathbb{I}_\alpha \mid w_\alpha(\tilde{\mathcal{E}}_{i+1}) \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1\}$	Def. 3.12
$E_\alpha(\rho)$	upper bound of $e(\mathcal{J})_\alpha$	(4.3)
$W_i = W_i(\lambda)$	$\{v \in W \mid s_i(v) = \dots = s_m(v) = 0\} \subset W$	(5.1)
$\omega_{\mathbf{a}}(\lambda)$	$\omega(\lambda) + \mu_{\mathbf{a}}(\lambda)$	(5.5)
$\Phi: \mathcal{H} \rightarrow \mathcal{C}$	Hilbert-Chow map	before Lem. 6.2
$\mathcal{K}, \tilde{\mathcal{K}} \subset \mathcal{H}$	slice polarized by $\omega_{\mathcal{H}/\mathcal{H}}(\mathbf{a} \cdot \mathbf{x})$	before (6.4)
$\tilde{\delta}(\mathcal{L})$	degree class for the line bundle \mathcal{L}	after (6.8)

2. CHOW STABILITY, CHOW WEIGHT AND NEWTON POLYGON

In this section, we first recall some basic facts from [16] on stability of a polarized curve; we then localize the calculation of the weight of $\text{Chow}(X)$ to a divisor on the

normalization of X , and interpret the contribution from each point of the divisor as the area of a generalized Newton polytope.

Throughout the paper, we fix a polarized (connected) curve $(X, \mathcal{O}_X(1))$, its associated embedding $\iota : X \rightarrow \mathbb{P}W$ (cf. (1.1)), and denote by $\text{Chow}(X)$ the Chow point of ι once and for all. We also assume that X is nodal unless otherwise is mentioned.

We will reserve the symbol λ for a 1-PS of $SL(W)$; for such λ , we diagonalize its action by choosing

$$(2.1) \quad \mathbf{s} = \{s_0, \dots, s_m\} \quad \text{a basis of } W^\vee$$

so that under its dual bases the action λ is given by

$$(2.2) \quad \lambda(t) := \text{diag}[t^{\rho_0}, \dots, t^{\rho_m}] \cdot t^{-\rho_{\text{ave}}}, \quad \rho_0 \geq \rho_1 \geq \dots \geq \rho_m = 0,$$

and $\rho_{\text{ave}} = \frac{1}{m+1} \sum \rho_i$. We will call \mathbf{s} a diagonalizing basis of λ .

In [16], Mumford introduced a subsheaf

$$(2.3) \quad \mathcal{J}(\lambda) = (t^{\rho_0} s_0, \dots, t^{\rho_m} s_m) \subset \mathcal{O}_{X \times \mathbb{A}^1}(1) := p_X^* \mathcal{O}_X(1)$$

generated by sections in the paranthesis, where $p_X : X \times \mathbb{A}^1 \rightarrow X$ is the projection. Let $e(\mathcal{J}(\lambda))$ be the normalized leading coefficient (abbreviate to n.l.c.) of the Hilbert-Samuel polynomial:

$$(2.4) \quad e(\mathcal{J}(\lambda)) = \text{n.l.c. } \chi(\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}(\lambda)^k)$$

Proposition 2.1 (Mumford). *The λ -weight Chow(X) is*

$$\omega(\lambda) = \frac{2 \deg X}{m+1} \sum_{i=0}^m \rho_i - e(\mathcal{J}(\lambda)).$$

In the following, when the 1-PS λ and its diagonalizing basis \mathbf{s} are understood, we will drop λ from $\mathcal{J}(\lambda)$ and abbreviate $\mathcal{J}(\lambda)$ to \mathcal{J} . Our first step is to lift the calculation of $e(\mathcal{J})$ ($= e(\mathcal{J}(\lambda))$) to the normalization of X :

$$\pi : \tilde{X} \longrightarrow X.$$

We let

$$(2.5) \quad \tilde{s}_i = \pi^* s_i \in \mathcal{O}_{\tilde{X}}(1) := \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}},$$

and let $\tilde{\mathcal{J}}$ be the pull-back of \mathcal{J} :

$$(2.6) \quad \tilde{\mathcal{J}} = (t^{\rho_0} \tilde{s}_0, \dots, t^{\rho_m} \tilde{s}_m) \subset \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(1) = \mathcal{O}_{\tilde{X}}(1) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}.$$

Like $e(\mathcal{J})$, we define $e(\tilde{\mathcal{J}}) = \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}^k)$. We have the following proposition whose proof will be given at the end of this section.

Proposition 2.2. *We have $e(\mathcal{J}) = e(\tilde{\mathcal{J}})$.*

This Proposition enables us to lift the evaluation of $e(\mathcal{J})$ to \tilde{X} . Our next step is to localize the evaluation of $e(\tilde{\mathcal{J}})$ to individual $q \in \tilde{X}$. In order to do that, let z be a uniformizing parameter of \tilde{X} at q ; let t be the standard coordinates of \mathbb{A}^1 . We denote by $\hat{\mathcal{O}}_{\tilde{X},q}$ the formal completion of the local ring $\mathcal{O}_{\tilde{X},q}$ at its maximal ideal. We fix an isomorphism of $\hat{\mathcal{O}}_{\tilde{X},q}$ -modules (the first isomorphism below):

$$(2.7) \quad \varphi_q : \mathcal{O}_{\tilde{X}}(1) \otimes_{\mathcal{O}_{\tilde{X}}} \hat{\mathcal{O}}_{\tilde{X},q} \cong \hat{\mathcal{O}}_{\tilde{X},q} \cong \mathbb{C}[[z]],$$

where the second isomorphism is induced by the choice of z .

Definition 2.3. Let $\tilde{s}_i \in H^0(\mathcal{O}_{\tilde{X}}(1))$ be as in (2.5). We define

$$(2.8) \quad v(\tilde{s}_i, q) = \text{the vanishing order of } \tilde{s}_i \text{ at } q;$$

in case $\tilde{s}_i \equiv 0$ near q , we define $v(\tilde{s}_i, q) = \infty$. We set

$$(2.9) \quad h(q) = \max\{i \mid v(\tilde{s}_i, q) \neq \infty\} \quad \text{and} \quad w(\tilde{\mathcal{J}}, q) = v(\tilde{s}_{h(q)}, q).$$

The quantity $w(\tilde{\mathcal{J}}, q)$ is the width of the polygon Δ_q associated to $\tilde{\mathcal{J}}$ (at q) to be defined later.

We now look at the image of $\tilde{\mathcal{J}}$ under $\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(1) \rightarrow \hat{\mathcal{O}}_{\tilde{X} \times \mathbb{A}^1, (q, 0)}$. We let

$$(2.10) \quad I_q = (z^{v(\tilde{s}_m, q)}, z^{v(\tilde{s}_{m-1}, q)} t^{\rho_{m-1}}, \dots, z^{v(\tilde{s}_0, q)} t^{\rho_0}) \subset R = \mathbb{C}[[z, t]].$$

By construction, φ_q induces an isomorphism

$$(2.11) \quad (\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}^k) \otimes_{\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}} \hat{\mathcal{O}}_{\tilde{X} \times \mathbb{A}^1, (q, 0)} \cong R/I_q^k.$$

Notice that the right hand side is not a finite module when $h(q) < m$. Since $t^{\rho_i} \cdot \varphi_q(\tilde{s}_i) \in t^{\rho_{h(q)}} R$ for all i , the map

$$t^{k \cdot \rho_{h(q)}} R / (I_q \cap t^{\rho_{h(q)}} R)^k \longrightarrow R/I_q^k$$

induced by the inclusion $t^{k \cdot \rho_{h(q)}} R \subset R$ is injective. This time the R -module on the left hand side is a finite module. We define

$$(2.12) \quad e(\tilde{\mathcal{J}})_q = \text{n.l.c. dim } t^{k \cdot \rho_{h(q)}} R / (I_q \cap t^{\rho_{h(q)}} R)^k + 2\rho_{h(q)} \cdot w(\tilde{\mathcal{J}}, q).$$

Lemma 2.4. We have the summation formula² $e(\tilde{\mathcal{J}}) = \sum_{q \in \tilde{X}} e(\tilde{\mathcal{J}})_q$.

We need some preparation to prove this Lemma. We begin with a geometric interpretation of the quantity $e(\tilde{\mathcal{J}})_q$. Let $I \subset \mathbb{C}[z_1, z_2]$ be a monomial ideal and let Γ be the set of exponents of monomials in I ; namely, I is the linear span of the monomials $\{x^\gamma \mid \gamma \in \Gamma\}$, where Γ is a subset of $(\mathbb{N} \cup \{0\})^2 \subset \mathbb{R}_{\geq 0}^2$. ($\mathbb{R}_{\geq 0}^2$ is the first quadrant of \mathbb{R}^2 —the xy -plane.)

We then form the *closed convex hull* $\text{Conv}(\mathbb{R}_{\geq 0}^2 + \Gamma)$ of $\mathbb{R}_{\geq 0}^2 + \Gamma$. We let $\bar{\Gamma} = \text{Conv}(\mathbb{R}_{\geq 0}^2 + \Gamma) \cap \mathbb{N}^2$; the integral closure \bar{I} of I is the ideal generated by $\{x^\gamma \mid \gamma \in \bar{\Gamma}\}$ [6, Ex4.23].

We let $\Delta(I)$ be the Newton polygon of I :

$$\Delta(I) = \mathbb{R}_{\geq 0}^2 - \text{Conv}(\mathbb{R}_{\geq 0}^2 + \Gamma) \subset \mathbb{R}_{\geq 0}^2.$$

Lemma 2.5. Let $|\Delta(I)|$ be the area of the $\Delta(I)$. Then

$$\dim \mathbb{C}[z_1, z_2]/I^k = |\Delta(I)| \cdot k^2 + O(k).$$

Proof. Since \bar{I} is the integral closure of I , by Briancon-Skoda theorem [13, Thm 9.6.26], $I^k \subset \bar{I}^k \subset I^{k-1}$ for k sufficiently large. Since $\dim I^{k-1}/I^k$ is bounded from above by a linear function in k , $\dim \mathbb{C}[z_1, z_2]/I^k = \dim \mathbb{C}[z_1, z_2]/\bar{I}^k + O(k)$.

Further, $\dim \mathbb{C}[z_1, z_2]/\bar{I}^k$ is precisely the number of lattice points in $k\Delta(\bar{I}) = k\Delta(I)$. From the work of Kantor and Khovanski [11, 5], the number of lattice points inside the polygon is given by $|\Delta(I)| \cdot k^2 + O(1)$. This proves the Lemma. \square

We now come back to the 1-PS λ and its diagonalizing basis $\mathbf{s} = \{s_i\}$.

Definition 2.6. For any $q \in \tilde{X}$, we define $\Gamma_q = \{(v(\tilde{s}_i, q), \rho_i)\}_{0 \leq i \leq m} \subset (\mathbb{N} \cup \{0\})^2$ and define the Newton polygon (of $\tilde{\mathcal{J}} = \tilde{\mathcal{J}}(\lambda)$) at q to be

$$\Delta_q(\lambda) := (\mathbb{R}_{\geq 0}^2 - \text{Conv}(\mathbb{R}_{\geq 0}^2 + \Gamma_q)) \cap ([0, w(\tilde{\mathcal{J}}, q)] \times \mathbb{R}_{\geq 0}).$$

² We were informed that similar formula was obtained by Swinarski in 2008.

We will abbreviate $\Delta_q(\lambda)$ to Δ_q when the choice of the basis \mathbf{s} is understood. Let $|\Delta_q|$ be the area of Δ_q .

Corollary 2.7. *We have $e(\tilde{\mathcal{J}})_q = 2|\Delta_q|$; henceforth, $e(\tilde{\mathcal{J}}) = 2 \sum_{q \in \tilde{X}} |\Delta_q|$.*

Proof. Since Δ_q is the union of $\Delta_q \cap [0, w(\tilde{\mathcal{J}}, q)] \times [\rho_{h(q)}, \infty)$ with $[0, w(\tilde{\mathcal{J}}, q)] \times [0, \rho_{h(q)}]$, by (2.4), (2.12) and Lemma 2.5,

$$e(\tilde{\mathcal{J}})_q = 2 \cdot |\Delta_q \cap [0, w(\tilde{\mathcal{J}}, q)] \times [\rho_{h(q)}, \infty)| + 2 \cdot \rho_{h(q)} \cdot w(\tilde{\mathcal{J}}, q) = 2|\Delta_q|.$$

The second identity follows from Lemma 2.4. \square

This formula will be used to estimate the quantity $e(\mathcal{J}) = e(\tilde{\mathcal{J}})$ in the next section. For now, we prove Proposition 2.2.

Proof of Proposition 2.2. Let p_1, \dots, p_l be the nodes of X ; let $\xi = \pi \times 1_{\mathbb{A}^1} : \tilde{X} \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1$ be the projection. Tensoring the exact sequence

$$0 \rightarrow \mathcal{O}_{X \times \mathbb{A}^1} \rightarrow \xi_* \mathcal{O}_{\tilde{X} \times \mathbb{A}^1} \rightarrow \bigoplus_{j=1}^l \mathcal{O}_{p_j \times \mathbb{A}^1} \rightarrow 0$$

with $\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k$, we obtain an exact sequence

$$\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k \xrightarrow{f_k} (\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \xi_* \mathcal{O}_{\tilde{X} \times \mathbb{A}^1} \rightarrow \bigoplus_{\alpha=1}^r (\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k)|_{p_j \times \mathbb{A}^1} \rightarrow 0.$$

By projection formula, we have

$$\xi_* (\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}^k) = \xi_* (\xi^* (\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k)) = (\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \xi_* \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}.$$

Thus

$$e(\tilde{\mathcal{J}}) = \text{n.l.c.} \chi(\xi_* (\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}^k)) = \text{n.l.c.} \chi((\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \xi_* \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}),$$

which equals

$$\text{n.l.c.} (\chi(\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) - \dim \ker f_k + \sum_{i=1}^l \chi((\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k)|_{p_j \times \mathbb{A}^1})).$$

We claim that both

$$(2.13) \quad \chi((\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \mathcal{O}_{p_j \times \mathbb{A}^1}) \quad \text{and} \quad \dim \ker f_k$$

are linear in k . This will prove the Proposition.

We begin with the first claim. We let q be one of the node of X ; let q^+ and q^- be the preimages $\pi^{-1}(q) \subset \tilde{X}$, and let x and y be uniformizing parameters of \tilde{X} at q^+ and q^- , respectively. Then after fixing an isomorphism $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,q} \cong \mathcal{O}_{X,q}$ near q and denoting $R = \mathbb{k}[[x, y]]/(xy)$, we have isomorphism

$$(2.14) \quad (\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}^k) \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \mathcal{O}_{q \times \mathbb{A}^1} \cong (R[t]/I^k) \otimes_{R[t]} R[t]/(x, y),$$

where $I \subset R[t]$ is the ideal generated by $t^{\rho_i} \hat{s}_i$, $i = 0, \dots, m$, and \hat{s}_i are formal germs of s_i at q as elements in R . Since for some i the value $s_i(q) \neq 0$, $i_q = \max\{i \mid s_i(q) \neq 0\}$ is finite. Thus the right hand side of (2.14) is isomorphic to $R[t]/(I^k, x, y) = \mathbb{k}[t]/(t^{k \cdot i_q})$ whose dimension is linear in k . This proves the first claim.

For the second claim, since the kernel of f_k consists of torsion elements supported on the union of $p_1 \times \mathbb{A}^1, \dots, p_l \times \mathbb{A}^1$. Hence to prove the claim, we only need to study the kernel of an analogue homomorphism

$$\bar{f}_k : R[t]/I^k \rightarrow (R[t]/I^k) \otimes_{R[t]} (\mathbb{k}[[x]][t] \oplus \mathbb{k}[[y]][t]),$$

where I is as in the previous paragraph, and $R[t] \rightarrow \mathbb{k}[[x]][t] \oplus \mathbb{k}[[y]][t]$ is the normalization homomorphism $g(x, y, t) \mapsto (g(x, 0, t), g(0, y, t))$. Since the domain and the target of \bar{f}_k are t -graded rings and \bar{f}_k is a homomorphism of graded rings, as vector spaces

$$\ker \bar{f}_k = \bigoplus_{j \geq 0} \ker \{ (\bar{f}_k)_j : t^j R / (I^k \cap t^j R) \rightarrow (t^j R / (I^k \cap t^j R)) \otimes_R (\mathbb{k}[[x]] \oplus \mathbb{k}[[y]]) \}.$$

Because $R = \mathbb{k}[[x, y]]/(xy)$, as R -modules, $t^j R / (I^k \cap t^j R)$ is isomorphic to R/J for J one of the ideals in the list:

$$R, (0), (x^e), (y^e), (x^e, y^{e'}), (x^e + y^{e'}), \text{ where } e, e' \in \mathbb{N}.$$

One checks that for J of the first five kinds, $\ker(\bar{f}_k)_j = 0$; for J of the last kind, $\ker(\bar{f}_k)_j \cong \mathbb{k}$. Thus we always have $\dim \ker(\bar{f}_k)_j \leq 1$. On the other hand, since $s_{i_q}(q) \neq 0$, $t^{\rho_{i_q}} \in I$ and $t^{k\rho_{i_q}} \in I^k$. Thus $\ker(\bar{f}_k)_j = 0$ for $j \geq ki_q$. This proves that $\dim \ker \bar{f}_k$ is at most linear in k . This proves the Proposition. \square

Because of this Proposition, we will work over the normalization \tilde{X} of X subsequently. To avoid possible confusion, we will reserve “ \sim ” to denote the associated objects lifted to \tilde{X} . For instance, we will denote by X_1, \dots, X_r the irreducible components of X , and denote by $\tilde{X}_1, \dots, \tilde{X}_r$ their respective normalizations. For the sections $t^{\rho_i} s_i$ in \mathcal{J} , $t^{\rho_i} \tilde{s}_i$ are their lifts in $\tilde{\mathcal{J}} = \mathcal{J} \otimes_{\mathcal{O}_{X \times \mathbb{A}^1}} \mathcal{O}_{\tilde{X} \times \mathbb{A}^1}$. For consistence, we reserve subindex i for the sections s_i , and reserve the greek α for the index of the irreducible components $\{X_\alpha\}_{1 \leq \alpha \leq r}$.

Proof of Lemma 2.4. For each irreducible component $X_\alpha \subset X$, we let $\tilde{\mathcal{J}}_\alpha = \tilde{\mathcal{J}}|_{\tilde{X}_\alpha \times \mathbb{A}^1} \subset \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(1)$. Then

$$e(\tilde{\mathcal{J}}) = \sum_{\alpha=1}^r \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}_\alpha^k) = \sum_{\alpha=1}^r e(\tilde{\mathcal{J}}_\alpha).$$

Thus to prove the Lemma, we only need to show that for each X_α ,

$$e(\tilde{\mathcal{J}}_\alpha) = \sum_{q \in \tilde{X}_\alpha} e(\tilde{\mathcal{J}}_\alpha)_q,$$

where $e(\tilde{\mathcal{J}}_\alpha)_q = e(\tilde{\mathcal{J}})_q$ when $q \in \tilde{X}_\alpha$. To proceed, we notice that $h(q)$ (cf.(2.9)) is a locally constant function on \tilde{X}_α ; we let \tilde{h}_α be the values of $h(q)$ for $q \in \tilde{X}_\alpha$. Then we have

$$(2.15) \quad \tilde{h}_\alpha = \min_i \{i \mid \tilde{s}_j|_{\tilde{X}_\alpha} = 0, \text{ for } j \geq i+1\}.$$

Thus $t^{\rho_{h_\alpha}}$ divides $t^{\rho_i} \tilde{s}_i$ for all $i > \tilde{h}_\alpha$. Since $\rho_i \geq \rho_{i+1}$, the same division holds for all i . We let $\bar{\rho}_i = \rho_i - \rho_{\tilde{h}_\alpha}$, and introduce ideal

$$\tilde{\mathcal{R}}_\alpha = (t^{\bar{\rho}_0} \tilde{s}_0, t^{\bar{\rho}_1} \tilde{s}_1, \dots, t^{\bar{\rho}_{h_\alpha}} \tilde{s}_{h_\alpha}) \subset \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(1).$$

This way, $\tilde{\mathcal{J}}_\alpha = t^{\rho_{h_\alpha}} \tilde{\mathcal{R}}_\alpha \subset t^{\rho_{h_\alpha}} \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(1)$.

We let $(t^{k\rho_{h_\alpha}}) = t^{k\rho_{h_\alpha}} \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)$; it belongs to the exact sequence

$$0 \longrightarrow (t^{k\rho_{h_\alpha}})/\tilde{\mathcal{J}}_\alpha^k \longrightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}_\alpha^k \longrightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\rho_{h_\alpha}}) \longrightarrow 0.$$

Since $(t^{k\rho_{h_\alpha}})/\tilde{\mathcal{J}}_\alpha^k = t^{k\rho_{h_\alpha}} \cdot (\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k)$ and $\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k$ is a finite module, we have

$$\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}_\alpha^k) = \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k) + \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\rho_{h_\alpha}})).$$

Taking the n.l.c. of individual term, and using

$$\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\rho_{h_\alpha}})) = k\rho_{h_\alpha} \cdot \chi(\mathcal{O}_{\tilde{X}_\alpha}(k)) = k^2 \rho_{h_\alpha} \cdot \deg X_\alpha + O(k),$$

we obtain

$$(2.16) \quad e(\tilde{\mathcal{J}}_\alpha) = \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X} \times \mathbb{A}^1}(k)/\tilde{\mathcal{J}}_\alpha^k) = \text{n.l.c. } \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k) + 2\rho_{\tilde{h}_\alpha} \cdot \deg \tilde{X}_\alpha.$$

Next let $\{q_1, \dots, q_l\}$ be the support of $(\tilde{s}_{\tilde{h}_\alpha} = 0) \cap \tilde{X}_\alpha$. Following the convention in (2.11), we have an isomorphism

$$\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k \xrightarrow{\cong} \bigoplus_{a=1}^l t^{k \cdot \rho_{\tilde{h}_\alpha}} R / (I_{q_a} \cap t^{\rho_{\tilde{h}_\alpha}} R)^k,$$

induced by restricting to germs at q_a after multiplying $t^{k \cdot \rho_{\tilde{h}_\alpha}}$. Adding that

$$\deg \tilde{X}_\alpha = \dim \mathcal{O}_{\tilde{X}_\alpha}(1)/(\tilde{s}_{\tilde{h}_\alpha}) = \sum_{a=1}^l w(\tilde{\mathcal{J}}, q_a),$$

(2.16) gives us

$$e(\tilde{\mathcal{J}}_\alpha) = \sum_{a=1}^l \left(\text{n.l.c. } h^0(t^{k \cdot \rho_{\tilde{h}_\alpha}} R / (I_{q_a} \cap t^{\rho_{\tilde{h}_\alpha}} R)^k) + 2\rho_{\tilde{h}_\alpha} \cdot w(\tilde{\mathcal{J}}, q_a) \right) = \sum_{q \in \tilde{X}_\alpha} e(\tilde{\mathcal{J}})_q.$$

This proves the Lemma. \square

Finally, we give one example that will be used later.

Example 2.8. Let $\mathbf{s} = \{s_i\}$ be a basis of $H^0(\mathcal{O}_X(1))$; using weights $\rho_0 = 1 > \rho_1 = \dots = \rho_m = 0$ we form a 1-PS with diagonalizing basis $\mathbf{s} = \{s_i\}$:

$$\lambda = \text{diag}[t, 1, \dots, 1] \cdot t^{-\frac{1}{m+1}}.$$

Suppose $p = \{s_1 = \dots = s_m = 0\} \in X$ is a single point. Then $e(\mathcal{J}(\lambda)) = 1$ (resp. $= 2$) when p is a smooth point (resp. nodal point) of X . Hence

$$\omega(\lambda) = \begin{cases} \frac{2 \deg X}{m+1} - 2, & q \text{ is a nodal point;} \\ \frac{2 \deg X}{m+1} - 1, & q \text{ is a regular point.} \end{cases}$$

3. STAIRCASE ONE-PARAMETER SUBGROUPS

We begin with some conventions attached to a fixed 1-PS λ and its diagonalizing basis $\{s_0, \dots, s_m\}$. For simplicity, we denote

$$\mathbb{I} = \{0, 1, \dots, m\}.$$

For each $i \in \mathbb{I}$, we introduce subsheaves

$$(3.1) \quad \mathcal{E}_i = \mathcal{E}(\lambda)_i := (s_i, s_{i+1}, \dots, s_m) \subset \mathcal{O}_X(1);$$

they form a decreasing sequence of subsheaves. Similarly, we introduce $\mathcal{O}_{\tilde{X}}$ -submodules

$$\tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}(\lambda)_i := (\tilde{s}_i, \tilde{s}_{i+1}, \dots, \tilde{s}_m) \subset \mathcal{O}_{\tilde{X}}(1).$$

Definition 3.1. We call $i \in \mathbb{I}$ a base index if $i = \tilde{h}_\alpha$ (cf. (2.15)) for some irreducible component X_α . For each X_α , we define $\Lambda_\alpha(\lambda) = \{q \in X_\alpha \mid s_{\tilde{h}_\alpha}(q) = 0\}$; define $\Lambda(\lambda) = \bigcup_{\alpha=1}^r \Lambda_\alpha(\lambda)$; define $\tilde{\Lambda}_\alpha(\lambda) = \{p \in \tilde{X}_\alpha \mid \tilde{s}_{\tilde{h}_\alpha}(p) = 0\}$, and define $\tilde{\Lambda}(\lambda) = \bigcup_{\alpha=1}^m \tilde{\Lambda}_\alpha(\lambda)$.

In the following, for any sheaf of $\mathcal{O}_{\tilde{X}}$ -modules \mathcal{F} and $p \in \tilde{X}$, we denote $\mathcal{F}_p := \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}, p}$, the localization of \mathcal{F} at p . We remark that for any $p \in \tilde{X}_\alpha$, $\tilde{h}(p) = \tilde{h}_\alpha$ is the largest index i so that $(\tilde{\mathcal{E}}_i)_p \neq 0$.

Definition 3.2. For closed point $p \in \tilde{X}_\alpha \subset \tilde{X}$, we define

$$\delta(\tilde{s}_i, p) = \text{length}(\tilde{\mathcal{E}}_i / \tilde{\mathcal{E}}_{i+1})_p \text{ when } i+1 \leq h_\alpha(p); \quad \delta(\tilde{s}_i, p) = 0 \text{ otherwise.}$$

We define the increments of \tilde{s}_i , along \tilde{X}_α and \tilde{X} , be (cycles)

$$\text{inc}_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p) p \quad \text{and} \quad \text{inc}(\tilde{s}_i) = \sum_{\alpha} \text{inc}_\alpha(\tilde{s}_i);$$

we define their degrees be $\delta_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p)$ and $\delta(\tilde{s}_i) = \sum_{\alpha} \delta_\alpha(\tilde{s}_i)$. We also define the width of $\tilde{\mathcal{E}}_i$ at $p \in \tilde{X}_\alpha$ and at \tilde{X}_α for $i \leq h_\alpha$ be

$$(3.2) \quad w(\tilde{\mathcal{E}}_i, p) := \text{length}(\mathcal{O}_{\tilde{X}}(1) / \tilde{\mathcal{E}}_i)_p \quad \text{and} \quad w_\alpha(\tilde{\mathcal{E}}_i) := \sum_{p \in \tilde{X}_\alpha} w(\tilde{\mathcal{E}}_i, p).$$

We remark that for $p \in \tilde{X}_\alpha$, $i+1 \leq h(p)$ is equivalent to $(\tilde{\mathcal{E}}_{i+1})_p \neq 0$.

Definition 3.3. For any irreducible component $X_\alpha \subset X$ we introduce

$$(3.3) \quad \mathbb{I}_\alpha = \mathbb{I}_\alpha(\lambda) = \{i \in \mathbb{I} \mid \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \neq \emptyset \text{ or } i = h_\alpha\};$$

for $m_\alpha + 1 = |\mathbb{I}_\alpha|$, the order of \mathbb{I}_α , we introduce a re-indexing map

$$(3.4) \quad \text{ind}_\alpha : \mathbb{I}_\alpha \longrightarrow [0, m_\alpha] \cap \mathbb{Z}, \quad \text{order preserving and bijective.}$$

Similarly, for $p \in \tilde{X}$, we introduce

$$\mathbb{I}_p = \{i \in \mathbb{I} \mid p \in \text{inc}(\tilde{s}_i)\}.$$

For $m_p + 1 = |\mathbb{I}_p|$; we define similarly

$$\text{ind}_p : \mathbb{I}_p \longrightarrow [0, m_p] \cap \mathbb{Z}, \quad \text{order preserving and bijective.}$$

To define the staircase 1-PS, we need the following

Definition 3.4. For each \mathcal{E}_i , we define its codegree

$$(3.5) \quad \text{codeg}(\mathcal{E}_i) = \text{length}(\mathcal{O}_Y(1) / \mathcal{E}_i|_Y) + \deg \mathcal{O}_{Y^c}(1), \quad Y = \text{Supp}(\mathcal{E}_i),$$

where $\text{Supp}(\mathcal{E}_i)$ is the smallest closed subscheme $Y \subset X$ so that the tautological $\mathcal{E}_i \rightarrow \mathcal{E}_i|_Y := \mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is injective, and $Y^c = X \setminus Y$. Since \mathcal{E}_i is decreasing, $\text{codeg}(\mathcal{E}_i)$ is increasing.

Definition 3.5. We say a 1-PS λ is a semi-staircase after index i if for any $i < j \leq m$, either $\text{codeg}(\mathcal{E}_{j-1}) < \text{codeg}(\mathcal{E}_j)$, or $j = h_\alpha + 1$ (cf. (2.15)) for some irreducible component $X_\alpha \subset X$. We say λ is a semi-staircase when λ is a semi-staircase after index 1.

Proposition 3.6. Given a 1-PS λ , there is a semi-staircase 1-PS λ' so that $\omega(\lambda) \geq \omega(\lambda')$.

Proof. Suppose λ is a semi-staircase at index i but not at $i-1$ then

$$(3.6) \quad \text{codeg}(\mathcal{E}_{i-1}) = \text{codeg}(\mathcal{E}_i) < \text{codeg}(\mathcal{E}_{i+1}).$$

We claim that $\mathcal{E}_{i-1} = \mathcal{E}_i$. Since $\text{Supp}(\mathcal{E}_i)$ is always a subcurve of X and $i \neq h_\alpha + 1$ for all α by the assumption, Y is also the support of \mathcal{E}_{i-1} . Consequently, $\mathcal{E}_i|_Y \subset \mathcal{E}_{i-1}|_Y$ and $\mathcal{E}_{i-1}|_Y / \mathcal{E}_i|_Y$ is a finite module. Then $\text{codeg}(\mathcal{E}_{i-1}) = \text{codeg}(\mathcal{E}_i)$ implies that $\text{length}(\mathcal{E}_{i-1}|_Y / \mathcal{E}_i|_Y) = 0$. This proves that $\mathcal{E}_{i-1} = \mathcal{E}_i$.

As a consequence, we have $\mathcal{E}_{i-1} = \mathcal{E}_i \supsetneq \mathcal{E}_{i+1}$. Therefore, there is a point $p \in X$ such that if we denote by $\hat{s}_j \in \hat{\mathcal{O}}_{X,p}(1)$ the formal germ of s_j at p , then as $\hat{\mathcal{O}}_{X,p}$ -modules

$$(3.7) \quad \hat{\mathcal{O}}_{X,p}(1) \supset (\hat{s}_{i-1}, \dots, \hat{s}_m) = (\hat{s}_i, \dots, \hat{s}_m) \supsetneq (\hat{s}_{i+1}, \dots, \hat{s}_m).$$

By the middle equality, we can find $\hat{c}_j \in \hat{\mathcal{O}}_{X,p}$ such that $\hat{s}_{i-1} = \sum_{j=i}^m \hat{c}_j \hat{s}_j$.

We now construct a new basis \mathbf{s}' . Let $c = \hat{c}_i(p) \in \mathbb{k}$. We define

$$(3.8) \quad s'_j = s_j \quad \text{for } j \neq i, i-1; \quad s'_i = s_{i-1} - cs_i; \quad s'_{i-1} = s_i.$$

Clearly, $\mathbf{s}' = \{s'_i\}$ is a basis of $H^0(\mathcal{O}_X(1))$. For $j \neq i$, because the linear span of $\{s_j, \dots, s_m\}$ equals the linear span of $\{s'_j, \dots, s'_m\}$, we have $\mathcal{E}_j = \mathcal{E}'_j$, where \mathcal{E}'_j is the \mathcal{E}_i in (3.1) with s_i replaced by s'_i .

For i , we claim that $\mathcal{E}'_i \subsetneq \mathcal{E}_i$. The inclusion $\mathcal{E}'_i \subset \mathcal{E}_i$ follows from $\mathcal{E}'_i \subset \mathcal{E}_{i-1} = \mathcal{E}_i$. For the inequality, we claim that

$$(\hat{s}_{i-1} - c\hat{s}_i, \hat{s}_{i+1}, \dots, \hat{s}_m) \neq (\hat{s}_i, \hat{s}_{i+1}, \dots, \hat{s}_m).$$

Suppose instead the identity holds, then there are constants $a_j \in \mathbb{k}$ such that

$$\hat{s}_i = a_i(\hat{s}_{i-1} - c\hat{s}_i) + \sum_{j=i+1}^m a_j \hat{s}_j = (a_i(\hat{s}_{i-1} - \hat{c}_i \hat{s}_i) + \sum_{j=i+1}^m a_j \hat{s}_j) + a_i(\hat{c}_i - c)\hat{s}_i.$$

Combined with $\hat{s}_{i-1} = \sum_{j=i}^m \hat{c}_j \hat{s}_j$, we conclude that $\hat{s}_i \in (\hat{s}_{i+1}, \dots, \hat{s}_m) + \hat{s}_i \mathfrak{m}$, where $\mathfrak{m} \subset \hat{\mathcal{O}}_{X,p}$ is the maximal ideal. By Nakayama Lemma, $\hat{s}_i \in (\hat{s}_{i+1}, \dots, \hat{s}_m)$, contradicting to (3.7). This proves the claim.

Finally, we claim that if we define λ' be the 1-PS with diagonalizing basis \mathbf{s}' and associated weights $\{\rho\}_{i \in \mathbb{I}}$, then $\omega(\lambda') \leq \omega(\lambda)$. By Mumford's formula (cf. Prop. 2.1), this is equivalent to $e(\mathcal{J}(\lambda')) \geq e(\mathcal{J}(\lambda))$. By our construction, $\mathcal{E}'_i \subset \mathcal{E}_i$ for all $i \in \mathbb{I}$; hence since $\rho_{i-1} \geq \rho_i$, $\mathcal{J}(\lambda') \subset \mathcal{J}(\lambda)$. Thus $\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}(\lambda')^k$ surjects onto $\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}(\lambda)^k$. This proves $e(\mathcal{J}(\lambda')) \geq e(\mathcal{J}(\lambda))$.

So far, for any λ that is not a semi-staircase, we have constructed a new λ' so that $\mathcal{J}(\lambda') \subset \mathcal{J}(\lambda)$. We now claim that by continuing this process, we eventually arrive at a semi-staircase λ' . Suppose not, then we can constructed an infinite sequence of 1-PS

$$\lambda = \lambda_0, \lambda_1, \dots, \lambda_l, \dots$$

so that $\mathcal{E}(\lambda_{l+1})_i \subset \mathcal{E}(\lambda_l)_i$ for all $i \in \mathbb{I}$, and for some i , $\mathcal{E}(\lambda_{l+1})_i \neq \mathcal{E}(\lambda_l)_i$. (Here $\mathcal{E}(\lambda_l)_i$ is the sheaf \mathcal{E}_i in (3.1) with λ replaced by λ_l .) Because $\text{codeg}(\mathcal{E}(\lambda_l)_i) \leq \deg \mathcal{O}_X(1)$, for each i , the sequence

$$\mathcal{E}(\lambda_0)_i \subset \mathcal{E}(\lambda_1)_i \subset \dots \subset \mathcal{E}(\lambda_l)_i \subset \mathcal{E}(\lambda_{l+1})_i \subset \dots$$

stabilize at finite places. In particular, after finite place, we will have $\mathcal{E}(\lambda_l)_i = \mathcal{E}(\lambda_{l+1})_i$ for all i ; or equivalently, $\mathcal{J}(\lambda_l) = \mathcal{J}(\lambda_{l+1})$, a contradiction. This proves that this process eventually provides us a semi-staircase λ' such that $\omega(\lambda) \geq \omega(\lambda')$. \square

Remark 3.7. We remark that for a semi-staircase λ , the inclusions $\mathcal{O}_{\tilde{X}}(1) = \tilde{\mathcal{E}}_0 \supsetneq \tilde{\mathcal{E}}_1 \supsetneq \dots \supsetneq \tilde{\mathcal{E}}_m \neq 0$ are proper.

Definition 3.8. We say a semi-staircase 1-PS λ is a staircase if for any $p \in \tilde{\Lambda}$, $v(\tilde{s}_i, p) \leq v(\tilde{s}_{i+1}, p)$ for all i (cf. Definition 2.3).

Corollary 3.9. Proposition 3.6 holds with semi-staircase replaced by staircase.

Proof. By Proposition 2.1, the λ -weight $\omega(\lambda)$ (of $\text{Chow}(X)$) depends only the sheaf $\mathcal{J}(\lambda)$ and the weights $\{\rho_i\}$. Thus, for any 1-PS λ' with $\mathcal{J}(\lambda) = \mathcal{J}(\lambda')$ and having identical weights $\{\rho'_i\}$ as that of λ , we have $\omega(\lambda) = \omega(\lambda')$.

Given any 1-PS, we let λ be the corresponding semi-staircase constructed in Proposition 3.6. Let $\tilde{\Lambda}$ and $\{\tilde{s}_i\}$ be the associated objects of λ . Since $\tilde{\Lambda}$ is a finite set, if we replace s_i

by $s'_i = s_i + \sum_{j>i} c_{ij}s_j$ for a general choice of $c_{ij} \in \mathbb{C}$, the new 1-PS with the same $\{\rho_i\}$ but new basis $\{s'_i\}$ will be a desired *staircase* 1-PS. \square

Lemma 3.10. *Suppose λ is a staircase 1-PS, then for $p \in \tilde{X}_\alpha$ and $i < h_\alpha$, $w(\tilde{\mathcal{E}}_i, p) = v(\tilde{s}_i, p)$, and $\delta(\tilde{s}_{i-1}, p) = v(\tilde{s}_i, p) - v(\tilde{s}_{i-1}, p)$.*

Proof. The proof is a direct consequence of the definition of staircase 1-PS. \square

As we will see, if λ is a *staircase* 1-PS then for most of i , $\delta(\tilde{s}_i) = 1$. For those i with $\delta(\tilde{s}_i) > 1$, we will give a detailed characterization (cf. Prop. 3.11). To this purpose, for any subcurve $Y \subset X$, we denote by N_Y to be the set of *nodes* of X in Y ; namely, $N_Y = X_{\text{node}} \cap Y$. We denote (cf. (1.3))

$$(3.9) \quad L_Y := Y \cap Y^{\mathbb{G}},$$

and call it the *linking nodes* of Y . Moreover, let

$$(3.10) \quad \tilde{N}_Y := \pi^{-1}(N_Y) \cap \tilde{Y} \quad \text{and} \quad \tilde{L}_Y := \pi^{-1}(L_Y) \cap \tilde{Y} \subset \tilde{N}_Y.$$

Since we reserve α for the index of the components X_α , we abbreviate

$$(3.11) \quad N_\alpha := N_{X_\alpha}, \quad \tilde{N}_\alpha := \tilde{N}_{X_\alpha}, \quad L_\alpha := L_{X_\alpha}, \quad \tilde{L}_\alpha := \tilde{L}_{X_\alpha}, \quad \ell_\alpha := |L_\alpha|.$$

We now state a characterization of those indices with $\delta(\tilde{s}_i) > 1$.

Proposition 3.11. *Suppose λ is a staircase 1-PS. Let $i \in \mathbb{I}_\alpha$ be a non-base index (cf. Definition 3.1) and let $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$. Suppose $\delta(\tilde{s}_i) \geq 2$, and suppose further that either $\deg X_\alpha = 1$ or*

$$(3.12) \quad w_\alpha(\tilde{\mathcal{E}}_i) + 1 \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha,$$

then $q = \pi(p) \in X$ is a node of X , $\text{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$. When this happens, let $\{p, p'\} = \pi^{-1}(q)$ and let \tilde{X}_β be the component satisfying $p' \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\beta$ (possibly $\tilde{X}_\alpha = \tilde{X}_\beta$), and suppose further $\deg X_\beta > 1$ and

$$(3.13) \quad w_\beta(\tilde{\mathcal{E}}_i) + 1 \leq \deg X_\beta - 2g(X_\beta) - \ell_\beta,$$

then $\text{inc}(\tilde{s}_i) = p + p'$.

Before its proof, we introduce a few notations. Since \tilde{X}_α is smooth, we can view a zero-subscheme of \tilde{X}_α as a divisor as well. This way, the union of two effective divisors is the union as zero subschemes, and the sum is as sum of divisors. For example, $(\sum n_p p) \cup (\sum n'_p p) = \sum \max\{n_p, n'_p\} p$ and $(\sum n_p p) + (\sum n'_p p) = \sum (n_p + n'_p) p$.

Proof of Proposition 3.11. We will prove each part of the statement by repeatedly applying the following strategy. Suppose i satisfies (3.12) and $\delta(\tilde{s}_i) \geq 2$, we will construct a section $\zeta \in H^0(\mathcal{O}_X(1))$ so that the \mathcal{O}_X -modules $\mathcal{F}_j = (\zeta, s_j, \dots, s_m)$ fits into a strict filtration

$$(3.14) \quad \mathcal{F}_0 \supsetneq \dots \supsetneq \mathcal{F}_i \supsetneq \mathcal{F}_{i+1} \supsetneq \mathcal{E}_{i+1} \supsetneq \dots \supsetneq \mathcal{E}_m \neq 0.$$

Since \mathcal{E}_j and \mathcal{F}_j are generated by global sections of $H^0(\mathcal{O}_X(1))$, this implies $h^0(\mathcal{O}_X(1)) > m + 2$, a contradiction.

Let us assume $\deg X_\alpha > 1$ first, since for the case $\deg X_\alpha = 1$ the proof is rather easy. So $w_i(\tilde{\mathcal{E}}_i)$ satisfies (3.12). We recall an easy consequence of a vanishing result. Let $B \subset \tilde{X}_\alpha$ be a closed zero-subscheme such that

$$(3.15) \quad \deg B \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha + 1.$$

Let \tilde{N}_α be as defined in (3.11). We claim that the γ in the exact sequence

$$(3.16) \quad H^0(\mathcal{O}_{\tilde{X}_\alpha}(1)) \xrightarrow{\gamma} H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \longrightarrow H^1(\mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B))$$

is surjective. Indeed, using $\deg \tilde{N}_\alpha = 2g(X_\alpha) - 2g(\tilde{X}_\alpha) + \ell_\alpha$ and (3.15), we obtain

$$\deg \mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B) \geq \deg \tilde{X}_\alpha - \deg \tilde{N}_\alpha - \deg B \geq 2g(\tilde{X}_\alpha) - 1.$$

Therefore, the last term in (3.16) vanishes, which shows that the γ in (3.16) is surjective.

The section ζ mentioned before (3.14) will be chosen by picking an appropriate B and $v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1))$ so that any element $\tilde{\zeta}_\alpha \in \gamma^{-1}(v)$ descends to a section in $H^0(\mathcal{O}_{X_\alpha}(1))$ and the descent glue with $s_{i+1}|_{X_\alpha^\circ}$ to form a desired section ζ .

We let

$$(3.17) \quad \tilde{Z}_{\alpha,j} := (\tilde{s}_j = \cdots = \tilde{s}_m = 0) \cap \tilde{X}_\alpha \subset \tilde{X}_\alpha.$$

Since $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, $\delta_\alpha(\tilde{s}_i) \geq 1$. In case $\delta_\alpha(\tilde{s}_i) = 1$, we choose $B = \tilde{Z}_{\alpha,i} + p$, which is a subscheme of $\tilde{Z}_{\alpha,i+1}$. In case $\delta_\alpha(\tilde{s}_i) \geq 2$ and $\delta(\tilde{s}_i, p) = 1$, then there exists a $p' \neq p \in \tilde{X}_\alpha$ such that $p + p' \leq \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, (which is equivalent to $\tilde{Z}_{\alpha,i} + p + p' \subset \tilde{Z}_{\alpha,i+1}$). In case $\delta(\tilde{s}_i, p) \geq 2$, we choose $p' = p$. Combined, we let $B = \tilde{Z}_{\alpha,i} + p + p'$.

We then let

$$v_1 = \tilde{s}_{i+1}|_{\tilde{N}_\alpha} \in H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \quad \text{and} \quad v_2 \neq 0 \in H^0(\mathcal{O}_B(1)) \quad \text{s.t.} \quad v_2|_{B-p} = 0.$$

We claim that when $p \notin \tilde{N}_\alpha$, or $\text{ind}_p(i) \geq 1$, or $\delta(\tilde{s}_i, p) \geq 2$, then both $v_1|_{\tilde{N}_\alpha \cap B}$ and $v_2|_{\tilde{N}_\alpha \cap B}$ are zero.

Indeed, since $\tilde{N}_\alpha \cap B \subset \tilde{Z}_{\alpha,i+1}$ and $\tilde{s}_{i+1}|_{\tilde{Z}_{\alpha,i+1}} = 0$, we have $v_1|_{\tilde{N}_\alpha \cap B} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha \cap B} = 0$. For v_2 , we prove case by case. Suppose $p \notin \tilde{N}_\alpha$, then $\tilde{N}_\alpha \cap B = \tilde{N}_\alpha \cap (B - p)$; therefore since $v_2|_{B-p} = 0$, $v_2|_{\tilde{N}_\alpha \cap B} = 0$. Now suppose $p \in \tilde{N}_\alpha$. Since $v_2|_{B-p} = 0$, $v_2(\bar{p}) = 0$ for all $\bar{p} \in (\tilde{N}_\alpha \cap B) - \{p\}$. We remain to show that $v_2(p) = 0$. We write $B = \sum_{k=0}^l n_k p_k$, p_k distinct, as an effective divisor. Since $p \in B$, we can arrange $p_0 = p$. In case $\text{ind}_p(i) \geq 1$, we have $n_0 \geq 2$; in case $\delta(\tilde{s}_i, p) \geq 2$, since $p' = p$ we still have $n_0 \geq 2$. Thus $p \in B - p$ and $v_2(p) = 0$. This proves that v_1 and v_2 have identical images in $H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1))$. Consequently, (v_1, v_2) lifts to a section $v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1))$ using the exact sequence

$$H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \longrightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \oplus H^0(\mathcal{O}_B(1)) \longrightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1)).$$

Since $\deg B \leq w_\alpha(\tilde{\epsilon}_i) + 2$ and i satisfies (3.12) (, because we assume $\deg X_\alpha > 1$), $\deg B$ satisfies the inequality (3.15). Therefore, the γ in (3.16) is surjective. We let $\tilde{\zeta}_\alpha \in \gamma^{-1}(v) \subset H^0(\mathcal{O}_{\tilde{X}_\alpha}(1))$ be any lift. Because it is a lift of v_1 , $\tilde{\zeta}_\alpha|_{\tilde{N}_\alpha} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha}$. This implies that $\tilde{\zeta}_\alpha$ descends to a section $\zeta_\alpha \in H^0(\mathcal{O}_{X_\alpha}(1))$, and the descent ζ_α glues with $s_{i+1}|_{X_\alpha^\circ}$ to form a new section $\zeta \in H^0(\mathcal{O}_X(1))$.

We now prove the first part of the Proposition. We let $Z_{\alpha,j} \subset X_\alpha$ be the subscheme $Z_{\alpha,j} = (s_j = \cdots = s_m = 0) \cap X_\alpha$. We decompose $Z_{\alpha,j}$ into disjoint union $Z_{\alpha,j} = R_j \cup R'_j$ so that R_j is supported at $q = \pi(p)$ and R'_j is disjoint from q . We let $\tilde{Z}_\alpha = (\zeta = s_{i+1} = \cdots = s_m = 0) \cap X_\alpha$ and decompose $\tilde{Z}_\alpha = \tilde{R} \cup \tilde{R}'$ accordingly.

Suppose q is a smooth point of X . Then R_j and \tilde{R} are divisors, and can be written as $R_j = n_j q$ and $\tilde{R} = \bar{n} q$. In case $\delta_\alpha(\tilde{s}_i) = 1$, the choice of B ensures that $n_i = \bar{n} = n_{i+1} - 1$ and $R'_i \subset \tilde{R}' \subsetneq R'_{i+1}$. Thus

$$(s_i, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \subseteq (\zeta, s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \subsetneq (s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha}.$$

Further, since $\delta(\tilde{s}_i) \geq 2$ and $\zeta|_{X_\alpha^\mathfrak{c}} = s_{i+1}|_{X_\alpha^\mathfrak{c}}$, we have

$$(s_i, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\mathfrak{c}} \subsetneq (\zeta, s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\mathfrak{c}} \subseteq (s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\mathfrak{c}}.$$

Thus we have

$$(3.18) \quad \mathcal{E}_i \subsetneq \mathcal{F}_{i+1} \subsetneq \mathcal{E}_{i+1}.$$

In case $\delta_\alpha(\tilde{s}_i) \geq 2$, the choice of B ensures that $R_i \subsetneq \bar{R} \subsetneq R_{i+1}$. Thus

$$(s_i, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \subsetneq (\zeta, s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \subsetneq (s_{i+1}, \dots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha}.$$

This implies (3.18) as well. In summary, by the argument at the beginning of the proof, (3.18) leads to a contradiction which proves that q must be a node of X .

It remains to study the case where q is a node of X . A careful case by case study shows that when either $\text{ind}_p(i) \geq 1$ or $\delta(\tilde{s}_i, p) \geq 2$, then $Z_{\alpha, i} \subsetneq \bar{Z}_\alpha \subsetneq Z_{\alpha, i+1}$. Thus (3.18) holds, which leads to a contradiction. This proves that q is a node, $\text{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$.

We complete the proof of the first part by looking at the case $\deg X_\alpha = 1$. In this case $\text{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$, since otherwise $\deg X_\alpha = 1$ implies that $i = h_\alpha$, contradicting to the assumption that i is not a base index. We next show that $p \in L_\alpha$. But this is parallel to the proof of the case $\deg X_\alpha > 1$ by letting $B = p$ because $\delta_\alpha(\tilde{s}_i) = 1$. This completes the proof of the first part.

We now prove the further part. Let $\pi^{-1}(q) = \{p, p'\}$ with $p' \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\beta$ so that (3.13) holds. Then by the first part of the Proposition, we have $\text{ind}_p(i) = \text{ind}_{p'}(i) = 0$; hence $s_i(q) \neq 0$. Thus for $Z_j = (s_j = \dots = s_m = 0) \subset X$, we have $p \notin Z_i$ and $Z_{i+1} = p \cup S$, where S is a zero-subscheme disjoint from p . Since $Z_i \subsetneq Z_{i+1}$ and $p \notin Z_i$, we have $Z_i \subset S$. In case $Z_i = S$, then the further part of the Proposition holds. Suppose $Z_i \subsetneq S$, then repeating the proof of the first part of the Proposition, we can find a section $\zeta \in H^0(\mathcal{O}_X(1))$ so that $p \notin (\zeta = 0)$ and $S \subset (\zeta = 0)$. This way, we will have (3.18) again, which leads to a contradiction. This proves the further part of the Proposition. \square

The Proposition above motivates the following

Definition 3.12. For $\deg X_\alpha > 1$, we define the primary indices of X_α be

$$\mathbb{I}_\alpha^{\text{pri}} = \{i \in \mathbb{I}_\alpha \mid w_\alpha(\tilde{\mathcal{E}}_{i+1}) \leq \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1\},$$

for $\deg X_\alpha = 1$, we define $\mathbb{I}_\alpha^{\text{pri}} = \text{ind}_\alpha^{-1}(0) \subset \mathbb{I}_\alpha$. We say $i \in \mathbb{I}_\alpha$ is primary at $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$ if $i \in \mathbb{I}_\alpha^{\text{pri}}$; otherwise we say it is secondary. We define $\bar{j}_\alpha := \max\{i \mid i \in \mathbb{I}_\alpha^{\text{pri}}\}$.

Note that in the proof above, the assumption $\delta(\tilde{s}_i) \geq 2$ is used only to show that (3.14) is strict. If $i = h_\alpha$ for some α , then $\text{length}(\mathcal{E}_i/\mathcal{E}_{i+1}) = \infty$. This time we choose ζ so that $\mathcal{E}_i/\mathcal{F}_{i+1}$ is finite. Since $\mathcal{E}_i/\mathcal{E}_{i+1}$ is infinite, (3.14) remains strict. Hence we have

Proposition 3.13. Let $i = h_\alpha$ be a base index for some X_α , and let $p \in \text{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$. Suppose $\delta(\tilde{s}_i) \geq 1$ and $\deg X_\alpha = 1$, or $w_\alpha(\tilde{\mathcal{E}}_i)$ satisfies the inequality (3.12). Then $\text{ind}_p(i) = 0$, $\delta(\tilde{s}_i, p) = 1$, and $q = \tilde{\pi}(p) \in X_\alpha$ is a linking node of X_α . Further, let $\{p, p'\} = \pi^{-1}(q)$, then i must be secondary at p' (cf. Definition 3.12), and there is a component \tilde{X}_β so that $p' \in \tilde{X}_\beta$ and $i = h_\beta$.

Proof. The proof is parallel to the proof of the previous Proposition. We will omit it here. \square

Corollary 3.14. Denoting $w_\alpha^{\text{pri}} := w_\alpha(\tilde{\mathcal{E}}_{\bar{j}_\alpha+1})$, suppose $X_\alpha \subsetneq X$, then

$$(3.19) \quad 0 \leq \deg X_\alpha - w_\alpha^{\text{pri}} \leq 2(g(X_\alpha) + \ell_\alpha + 1).$$

Proof. The first inequality is trivial. We now prove the second one. If $\deg X_\alpha = 1$ we obtain $\deg X_\alpha - w_\alpha^{\text{pri}} = 0$, from which the second inequality trivially follows. So from now on we assume $\deg X_\alpha > 1$. We let $\bar{i} \in \mathbb{l}_\alpha$ be the index succeeding \bar{j}_α ; namely, \bar{i} is the smallest index $> \bar{j}_\alpha$ so that $\delta_\alpha(\tilde{s}_{\bar{i}}) \geq 1$. In particular, this implies that

$$(3.20) \quad \delta_\alpha(\tilde{s}_{\bar{j}_\alpha}) = \cdots = \delta_\alpha(\tilde{s}_{\bar{i}-1}) = 0.$$

Since $\bar{i} \notin \mathbb{l}_\alpha^{\text{pri}}$,

$$(3.21) \quad w_\alpha^{\text{pri}} = w_\alpha(\tilde{\mathcal{E}}_{\bar{j}_\alpha+1}) = w_\alpha(\tilde{\mathcal{E}}_{\bar{i}+1}) - \delta_\alpha(\tilde{s}_{\bar{i}}) > \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1 - \delta_\alpha(\tilde{s}_{\bar{i}}).$$

Thus when $\delta_\alpha(\tilde{s}_{\bar{i}}) \leq 2$, the second inequality follows from $\ell_\alpha \geq 1$ (, since $X_\alpha \subsetneq X$).

Suppose $\delta_\alpha(\tilde{s}_{\bar{i}}) > 2$. By our assumption \bar{i} is the index in \mathbb{l}_α immediately succeeding \bar{j}_α , we have $w_\alpha(\tilde{\mathcal{E}}_{\bar{i}}) = w_\alpha(\tilde{\mathcal{E}}_{\bar{j}_\alpha+1})$ because of (3.20). By Definition 3.12, $w_\alpha(\tilde{\mathcal{E}}_{\bar{i}})$ satisfies (3.12). So we can apply Proposition 3.11 to the index \bar{i} to conclude that every $p \in \text{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha$ lies in \tilde{N}_α and has $\delta(\tilde{s}_{\bar{i}}, p) = 1$.

We claim that $\text{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha$. Indeed, let $p \in \text{inc}(\tilde{s}_{\bar{i}}) \cap (\tilde{N}_\alpha \setminus \tilde{L}_\alpha)$, then the second part of Proposition 3.11 implies that $\text{inc}(\tilde{s}_{\bar{i}}) = p + p'$ and $\delta(\tilde{s}_{\bar{i}}) = 2$, contradicting to the assumption $\delta_\alpha(\tilde{s}_{\bar{i}}) > 2$. This proves that $\text{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha$. Adding that $\delta(\tilde{s}_{\bar{i}}, p) = 1$ for $p \in \text{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha$, we conclude that $\delta_\alpha(\tilde{s}_{\bar{i}}) \leq \ell_\alpha$. These and (3.21) proves the second inequality in (3.19). \square

4. MAIN ESTIMATE FOR IRREDUCIBLE CURVES

Throughout this section, we fix a staircase 1-PS λ , and an irreducible X_α . We will derive a sharp estimate of $e(\tilde{\mathcal{J}}_\alpha(\lambda))$ for the $X_\alpha \subset X$.

We let g_α be the genus of X_α ; we define the set of *special points*

$$(4.1) \quad \tilde{S}_\alpha = (\pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha) \cup \tilde{N}_\alpha \subset \tilde{X}_\alpha,$$

where $\mathbf{x} = (x_1, \dots, x_n) \subset X$ is the set of weighted points. We continue to denote by $\bar{\rho}_i = \rho_i - \rho_{\bar{h}_\alpha}$. For each $p \in \tilde{L}_\alpha$, we define the *initial index*

$$(4.2) \quad i_0(p) := \min\{i \mid i \in \mathbb{l}_p\}.$$

For $\deg X_\alpha > 1$ and a fixed $\epsilon > 0$, we define

$$(4.3) \quad E_\alpha(\rho) := \left(2 + \frac{2\epsilon}{\deg X_\alpha}\right) \sum_{i \in \mathbb{l}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \bar{\rho}_i - \left(1 + \frac{2\epsilon}{\deg X_\alpha}\right) \sum_{q \in \tilde{S}_\alpha \cap \tilde{L}_\alpha} \bar{\rho}_{i_0(q)} + 2 \deg X_\alpha \cdot \rho_{\bar{h}_\alpha};$$

for $\deg X_\alpha = 1$ satisfying $\mathbf{x} \cap X_\alpha = \emptyset$, we define

$$(4.4) \quad E_\alpha(\rho) := \delta_\alpha(\tilde{s}_{i_0}) \bar{\rho}_{i_0} + 2 \cdot \rho_{\bar{h}_\alpha}; \quad i_0 = \text{ind}_\alpha^{-1}(0).$$

It is clear that in both cases $E_\alpha(\rho)$ are linear in $\rho \in \mathbb{R}_+^{m+1}$. Our main result of this section is the following

Theorem 4.1. *For any $1 \geq \epsilon > 0$ there is a constant M depending only on g_α , ℓ_α and ϵ such that whenever $\deg X_\alpha \geq M$, then*

$$e(\tilde{\mathcal{J}}_\alpha(\lambda)) \leq E_\alpha(\rho).$$

In case $\deg X_\alpha = 1$ and $\mathbf{x} \cap X_\alpha = \emptyset$, the same inequality holds for $E_\alpha(\rho)$ defined in (4.4).

Note that the theorem implies that we can bounded $e(\tilde{\mathcal{J}}(\lambda))$ in terms of the primary ρ_i 's only. And for primary indices, we have a complete understanding of the multiplicity $\delta_\alpha(\tilde{s}_i)$ due to the detailed study in the previous section.

We begin with the following bound on the area of Δ_p in terms of $\{\rho_i\}$.

Lemma 4.2. *Let λ be a staircase. Then for each $p \in \tilde{\Lambda}_\alpha$ and any $0 \leq l \leq k \leq h_\alpha$, we have*

$$(4.5) \quad |\Delta_p \cap ([w(\tilde{\mathcal{E}}_l, p), w(\tilde{\mathcal{E}}_k, p)] \times \mathbb{R})| - \rho_{h_\alpha} (w(\tilde{\mathcal{E}}_k, p) - w(\tilde{\mathcal{E}}_l, p)) \leq \sum_{i \in \mathbb{I}_p \cap [l, k-1]} \delta(\tilde{s}_i, p) \bar{\rho}_i - \frac{(\bar{\rho}_{i_{\min}(p)} + \bar{\rho}_{i_{\max}(p)})}{2},$$

where $i_{\min} := \min(\mathbb{I}_p \cap [l, k-1])$ and $i_{\max} := \max(\mathbb{I}_p \cap [l, k-1])$.

Note that by letting $l = 0$ and $k = h_\alpha$, we obtain

$$(4.6) \quad |\Delta_p| - \rho_{h_\alpha} w(\tilde{\mathcal{E}}_{h_\alpha}, p) \leq \sum_{i \in \mathbb{I}_p} \delta(\tilde{s}_i, p) \bar{\rho}_i - \frac{\bar{\rho}_{i_0(p)}}{2}.$$

Proof. First, we notice that the above inequality is invariant when varying ρ_{h_α} , thus to prove the Lemma we can and do assume now on that $\rho_{h_\alpha} = 0$; hence $\bar{\rho}_i = \rho_i$.

Let $\Gamma_p := \{(w(\tilde{\mathcal{E}}_i, p), \rho_i)\}_{0 \leq i \leq m}$; it follows from Definition 2.6 and 3.8 that

$$(4.7) \quad \Delta_p = (\mathbb{R}_+^2 - \text{Conv}(\mathbb{R}_+^2 + \Gamma_p)) \cap ([0, w(\tilde{\mathcal{J}}, p)] \times \mathbb{R}).$$

Fixing an indexing

$$(4.8) \quad \mathbb{I}_p = \{i_0(p), \dots, i_d(p)\} \subset \mathbb{I}, \quad i_j(p) \text{ increasing and } d+1 = |\mathbb{I}_p|,$$

we let \mathbb{T} be the continuous piecewise linear function on $[0, w(\tilde{\mathcal{J}}, p)]$ defined by linear interpolating the points

$$\{(0, \rho_{i_0}), \dots, (w(\tilde{\mathcal{E}}_{i_k}, p), \rho_{i_k}), \dots, (w(\tilde{\mathcal{E}}_{i_d}, p), \rho_{h_\alpha})\} \subset \mathbb{R}^2,$$

and let $\Delta_{\mathbb{T}}$ be the polygon bounded on two sides by $x = 0$ and $x = w(\tilde{\mathcal{E}}_k, p)$, from below by $y = 0$ and from above by the graph of $y = \mathbb{T}$. By the convexity of Δ_p , we have

$$\Delta_p \cap ([w(\tilde{\mathcal{E}}_l, p), w(\tilde{\mathcal{E}}_k, p)] \times \mathbb{R}) \subset \Delta_{\mathbb{T}} \cap ([w(\tilde{\mathcal{E}}_l, p), w(\tilde{\mathcal{E}}_k, p)] \times \mathbb{R}) \subset \mathbb{R}^2.$$

By Lemma 3.10, $w(\tilde{\mathcal{E}}_i, p) = \sum_{j=0}^{i-1} \delta(\tilde{s}_j, p)$; hence

$$\begin{aligned} |\Delta_p \cap ([w(\tilde{\mathcal{E}}_l, p), w(\tilde{\mathcal{E}}_k, p)] \times \mathbb{R})| &\leq |\Delta_{\mathbb{T}} \cap ([w(\tilde{\mathcal{E}}_l, p), w(\tilde{\mathcal{E}}_k, p)] \times \mathbb{R})| \\ &\leq \sum_{i \in \mathbb{I}_p \cap [l, k-1]} \delta(\tilde{s}_i, p) \rho_i - \frac{1}{2}(\rho_{i_{\min}(p)} + \rho_{i_{\max}(p)}). \end{aligned}$$

This proves the Lemma. \square

With this lemma in hand, we now explain the key ingredient in the proof of the theorem. We will divide our estimates into two cases according to the size of $|\tilde{\Lambda}_\alpha|$ (cf. Definition 3.1). When $|\tilde{\Lambda}_\alpha|$ is large, applying Lemma 4.2, we will gain a sizable multiple of $\frac{1}{2}\rho_{i_0(p)}$'s (cf. (4.6)) in the estimate of Δ_p ; these extra gains will take care of the contributions from non-primary ρ_i 's. When $|\tilde{\Lambda}_\alpha|$ is small, one large Δ_p is sufficient to cancel the contribution from the non-primary ρ_i 's.

We need a few more notions. For any $p \in \tilde{\Lambda}_\alpha$, we let $\mathbb{I}_p^{\text{pri}} := \mathbb{I}_\alpha^{\text{pri}} \cap \mathbb{I}_p$, and define

$$(4.9) \quad \bar{j}_p := \max\{i \in \mathbb{I}_p^{\text{pri}}\}, \quad w^{\text{pri}}(p) := w(\tilde{\mathcal{E}}_{\bar{j}_p+1}, p), \quad \text{and} \quad w(p) := w(\tilde{\mathcal{J}}, p) \text{ (cf. (2.9))}.$$

Note that $w(p)$ is the base-width of the Newton polygon Δ_p . Using \bar{j}_p , we truncate the Newton polygon Δ_p by intersecting it with the strip $[0, w^{\text{pri}}(p)] \times \mathbb{R}$:

$$\Delta_p^{\text{pri}} := \Delta_p \cap [0, w^{\text{pri}}(p)] \times \mathbb{R}.$$

Our next Lemma says that if one Δ_p is big enough, the contribution from non-primary ρ_i 's will be absorbed by the difference between $E_\alpha(\rho)$ and $e(\tilde{\mathcal{J}}_\alpha(\rho))$.

Lemma 4.3. *For any $1 > \epsilon > 0$, there is an M depending only on ϵ , g_α and ℓ_α such that whenever $w(p) \geq M$ (cf.(4.9)),*

$$|\Delta_p^{\text{pri}}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\bar{\rho}_{\bar{\mathcal{J}}_\alpha} \leq \left(1 + \frac{\epsilon}{w(p)}\right) \sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\bar{\rho}_i + \rho_{\tilde{h}_\alpha} \cdot w^{\text{pri}}(p) - \left(\frac{1}{2} + \frac{\epsilon}{w(p)}\right)\bar{\rho}_{i_0(p)},$$

where w_α^{pri} is defined in Corollary 3.14.

Proof. By the same reason as in the proof of Lemma 4.2, we only need to treat the case that $\rho_{\tilde{h}_\alpha} = 0$; hence $\bar{\rho}_i = \rho_i$.

Our proof relies on the proximity of $\partial^+ \Delta_p$ ($\partial^+ \Delta_p$ is the boundary component of Δ_p lying in the (open) first quadrant) with the lattice points $(w(\tilde{\mathcal{E}}_i, p), \rho_i)$ (cf. (3.2)). In case they differ slightly, then the term $\frac{\epsilon}{w(p)} \sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\bar{\rho}_i$ is sufficient to absorb the term $2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{\mathcal{J}}_\alpha}$ in the inequality (note $\bar{\rho}_i = \rho_i$ by assumption). Otherwise, the difference between $\sum_{i \in \mathbb{I}_p^{\text{pri}} \cap [c, \bar{\mathcal{J}}_p]} \delta(\tilde{s}_i, p)\bar{\rho}_i$ (for some c that will be specified below) and $|\Delta_p|$ is sufficient to imply the desired estimate.

Let us assume $M > 4$, then $w(p) - \sqrt{w(p)} \geq 2$ whenever $w(p) \geq M$. We introduce

$$c = \max\{i \in \mathbb{I}_p^{\text{pri}} \mid (w(\tilde{\mathcal{E}}_i, p), \frac{\rho_i}{2}) \in \Delta_p \subset \mathbb{R}^2\} \quad \text{and} \quad w^c(p) := w(\tilde{\mathcal{E}}_c, p),$$

and let

$$\Delta_p^{\leq c} = \Delta_p \cap [0, w^c(p)] \times \mathbb{R}.$$

We divide our study into two cases. The first is when $w(p) - w^c(p) \leq \sqrt{w(p)}$. We introduce trapezoid Θ to be the region between x -axis and the line passing through the points $(w(p), 0)$ and $(w^c(p), \frac{\rho_c}{2})$ intersecting with the strip $[1, w^c(p)] \times \mathbb{R}$. Then by our assumption

$$w^c(p) - 1 \geq w(p) - \sqrt{w(p)} - 1 \geq \frac{w(p) - \sqrt{w(p)}}{2}.$$

Since the length of the two vertical edges of Θ are of $\frac{\rho_c}{2}$ and $\frac{w(p)-1}{w(p)-w^c(p)} \cdot \frac{\rho_c}{2}$, we deduce

$$\begin{aligned} |\Theta| &= \left(\frac{w(p)-1}{w(p)-w^c(p)} + 1\right)(w^c(p)-1)\frac{\rho_c}{4} \\ &\geq \left(\frac{\sqrt{w(p)}}{2} + 1\right) \cdot \frac{w(p) - \sqrt{w(p)}}{2} \cdot \frac{\rho_c}{4} \geq \frac{w(p)^{3/2} \cdot \rho_c}{32}. \end{aligned}$$

Since the piecewise linear $\partial^+ \Delta_p$ is convex, Θ lies inside Δ_p , hence

$$|\Delta_p| - \frac{\rho_{i_0(p)}}{2} > |\Theta| > \frac{w(p)^{3/2}}{32} \rho_c.$$

By the definition of Δ_p^{pri} , the difference between the base-width of Δ_p^{pri} and of Δ_p is bounded by $w(p) - w^{\text{pri}}(p)$; therefore by Lemma 4.2 we have

$$\begin{aligned} |\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{\mathcal{J}}_\alpha} &\geq |\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} + (w(p) - w^{\text{pri}}(p))\rho_{\bar{\mathcal{J}}_\alpha} \\ &\geq |\Delta_p| - \frac{\rho_{i_0(p)}}{2} \geq \frac{w(p)^{3/2}}{32} \rho_c. \end{aligned}$$

Since $\rho_{\bar{j}_\alpha} \leq \rho_c$, this implies

$$(4.10) \quad |\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} > \left(\frac{w(p)^{3/2}}{32} - 2(\deg X_\alpha - w_\alpha^{\text{pri}}) \right) \rho_c .$$

We now choose M so that $M^{3/2} \geq 2^8(g_\alpha + \ell_\alpha + 1)$. By Corollary 3.14, we have $\deg X_\alpha - w_\alpha^{\text{pri}} \leq 2(g_\alpha + \ell_\alpha + 1)$. Therefore when $w(p) \geq M$, we have

$$2(\deg X_\alpha - w_\alpha^{\text{pri}}) \leq 4(g_\alpha + \ell_\alpha + 1) \leq \frac{w(p)^{3/2}}{64} .$$

Plugging this into (4.10), we obtain

$$|\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} > \frac{w(p)^{3/2}}{64} \rho_c \quad , \text{ equivalently, } \quad \rho_c \leq \frac{2^6}{w(p)^{3/2}} \left(|\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} \right) ,$$

hence

$$2(\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\bar{j}_\alpha} \leq 2(\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_c \leq \frac{2^6(\deg X_\alpha - w_\alpha^{\text{pri}})}{w(p)^{3/2}} \left(|\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} \right) .$$

So if we assume further $M \geq 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2$, then whenever $w(p) \geq M$ we have $2^6(\deg X_\alpha - w_\alpha^{\text{pri}})w(p)^{-3/2} \leq \epsilon/w(p)$, thus

$$\begin{aligned} |\Delta_p^{\text{pri}}| - \frac{1}{2}\rho_{i_0(p)} + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{j}_\alpha} &\leq \left(1 + \frac{\epsilon}{w(p)}\right) \left(|\Delta_p^{\text{pri}}| - \frac{\rho_{i_0(p)}}{2} \right) \\ &\leq \left(1 + \frac{\epsilon}{w(p)}\right) \left(\sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p) \rho_i - \rho_{i_0(p)} \right), \end{aligned}$$

where the last inequality follows from Lemma 4.2. This proves the Lemma in this case.

The second case is when $w(p) - w^c(p) > \sqrt{w(p)}$. By the definition of c , for $j \in \mathbb{J} := \mathbb{I}_p \cap (c, \bar{j}_\alpha]$, $(w(\tilde{\mathcal{E}}_j, p), \rho_j/2) \notin \Delta_p$. Since $\partial^+ \Delta_p$ is convex, by Lemma 4.2, we have

$$\sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) \rho_i - |\Delta_p^{\text{pri}} \setminus \Delta_p^{\leq c}| \geq \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) \rho_i / 2 .$$

Notice that

$$\begin{aligned} \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) &= w^{\text{pri}}(p) - w^c(p) = w(p) - w^c(p) - (w(p) - w^{\text{pri}}(p)) \\ &> \sqrt{w(p)} - (\deg X_\alpha - w_\alpha^{\text{pri}}) , \end{aligned}$$

since $\deg X_\alpha - w_\alpha^{\text{pri}} \geq w(p) - w^{\text{pri}}(p)$ and $w(p) - w^c(p) > \sqrt{w(p)}$ by our assumption. If we choose

$$M \geq 10^2(g_\alpha + \ell_\alpha + 1)^2 \geq 5^2(\deg X_\alpha - w_\alpha^{\text{pri}})^2$$

and require $w(p) \geq M$, then $\sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) \geq 4(\deg X_\alpha - w_\alpha^{\text{pri}})$. This implies

$$\sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) \rho_i - |\Delta_p^{\text{pri}} \setminus \Delta_p^{\leq c}| \geq \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p) \rho_i / 2 \geq 2(\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\bar{j}_\alpha} ,$$

and combined with Lemma 4.2, we obtain

$$\begin{aligned}
|\Delta_p^{\text{pri}}| &+ 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{j}_\alpha} \\
&\leq |\Delta_p^{\leq c}| + |\Delta_p^{\text{pri}} \setminus \Delta_p^{\leq c}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{j}_\alpha} \\
&\leq |\Delta_p^{\leq c}| + |\Delta_p^{\text{pri}} \setminus \Delta_p^{\leq c}| - \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p)\rho_i + \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p)\rho_i + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{j}_\alpha} \\
&\leq |\Delta_p^{\leq c}| + \sum_{i \in \mathbb{J}} \delta(\tilde{s}_i, p)\rho_i \leq \sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\rho_i - \frac{\rho_{i_0(p)}}{2} \\
&< \left(1 + \frac{\epsilon}{w(p)}\right) \left(\sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\rho_i - \frac{\rho_{i_0(p)}}{2}\right) + \frac{\rho_{i_0(p)}}{2}.
\end{aligned}$$

In the end, since $\epsilon < 1$ we choose $M := 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2$. Then for $w(p) > M$, (4.3) holds. This proves the Lemma. \square

Proof of Theorem 4.1. First, by the same reason as in the proof of Lemma 4.3 we only need to deal with the case $\rho_{h_\alpha} = 0$ and $\bar{\rho}_i = \rho_i$. Also, when $\deg X_\alpha = 1$, then the statement is a direct consequence of Lemma 4.2. So from now on we assume that $\deg X_\alpha > 1$. Let $1 > \epsilon > 0$ be any constant. Since $\epsilon < 1$, we have $\frac{\epsilon}{\deg X_\alpha} \leq 1/2$ whenever $\deg X \geq M \geq 2$. We define σ to be the number of Newton polytopes supported on \tilde{X}_α . We divide our study into two cases.

The first case is when $\sigma > 10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|$. Since Corollary 3.14 implies

$$|\{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \mid i_0(p) > \bar{j}_\alpha\}| \leq \sum_{i \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \leq (\deg X_\alpha - w_\alpha^{\text{pri}}) \leq 2(g_\alpha + \ell_\alpha + 1),$$

the number of $p \in \tilde{\Lambda}_\alpha \setminus \tilde{S}_\alpha$ satisfying $\rho_{i_0(p)} \geq \rho_{\bar{j}_\alpha}$ is at least $8(\tilde{g}_\alpha + \ell_\alpha + 1)$. By Lemma 4.2, for each $p \in \tilde{\Lambda}_\alpha$, we gain an extra $\rho_{i_0(p)}/2$ on the right hand side in the estimate Δ_p in terms of $\{\rho_i\}_{i=0}^m$. This implies

$$(4.11) \quad \sum_{i \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i \leq (\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\bar{j}_\alpha} \leq 2(g_\alpha + \ell_\alpha + 1)\rho_{\bar{j}_\alpha} \leq \frac{1}{4} \sum_{p \in \tilde{\Lambda}_\alpha \setminus \tilde{S}_\alpha} \rho_{i_0(p)}.$$

So we obtain, via using Lemma 4.2 and summing over $p \in \tilde{\Lambda}_\alpha$,

$$\begin{aligned}
\sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p| &\leq \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i + \sum_{i \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \\
&= \left(\sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i - \frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \right) + \\
(4.12) \quad &+ \left(\sum_{i \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i + \frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \setminus \tilde{S}_\alpha} \rho_{i_0(p)} \right).
\end{aligned}$$

Using (4.11) and

$$\frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} \leq \sum_{i \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i \leq \frac{1}{4} \sum_{p \in \tilde{\Lambda}_\alpha \setminus \tilde{S}_\alpha} \rho_{i_0(p)},$$

the sum in the line of (4.12) is non-positive. Therefore, for any $0 < \epsilon < 1$ we have

$$\begin{aligned} \sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p| &\leq \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i - \frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} - \frac{1}{2} \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} \\ &\leq \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i - \left(\frac{1}{2} + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha} \rho_{i_0(p)} = \frac{E_\alpha(\rho)}{2}, \end{aligned}$$

since

$$\frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} \leq \frac{\epsilon}{\deg X_\alpha} \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i.$$

This verifies the Theorem in this case.

The other case is when $\sigma \leq 10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|$. By the pigeon hole principle, there exists at least one $p_0 \in \tilde{\Lambda}_\alpha$ such that

$$(4.13) \quad w(\tilde{\mathcal{J}}, p_0) \geq \frac{\deg X_\alpha}{\sigma} \geq \frac{\deg X_\alpha}{10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|}.$$

By Corollary 2.7, we have

$$\frac{e_{X_\alpha}(\mathcal{J}(\lambda))}{2} = \sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p|.$$

Our assumption $\epsilon \leq 1$, $1/\deg X \leq 1/2$ and Corollary 3.14 imply

$$(4.14) \quad \left(\frac{1}{2} + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} \leq \sum_{\substack{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} \leq (\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\tilde{\mathcal{J}}_\alpha}.$$

So we obtain

$$\begin{aligned} &\frac{e_{X_\alpha}(\mathcal{J}(\lambda))}{2} - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\ &= |\Delta_{p_0}^{\text{pri}}| + |\Delta_{p_0} \setminus \Delta_{p_0}^{\text{pri}}| + \sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} (|\Delta_p^{\text{pri}}| + |\Delta_p \setminus \Delta_p^{\text{pri}}|) - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2}. \end{aligned}$$

By Lemma 4.2 and the first inequality of (4.11), we have

$$|\Delta_{p_0} \setminus \Delta_{p_0}^{\text{pri}}| + \sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} |\Delta_p \setminus \Delta_p^{\text{pri}}| = \sum_{p \in \tilde{\Lambda}_\alpha} |\Delta_p \setminus \Delta_p^{\text{pri}}| \leq (\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\tilde{\mathcal{J}}_\alpha}.$$

So

$$\begin{aligned} &\frac{e_{X_\alpha}(\mathcal{J}(\lambda))}{2} - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\ &\leq |\Delta_{p_0}^{\text{pri}}| + (\deg X_\alpha - w_\alpha^{\text{pri}}) \rho_{\tilde{\mathcal{J}}_\alpha} + \sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} |\Delta_p^{\text{pri}}| - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \end{aligned}$$

$$\begin{aligned}
(4.15) \quad & \leq |\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\tilde{J}_\alpha} - \frac{\rho_{i_0(p_0)}}{2}|\{p_0\} \cap \tilde{S}_\alpha| + \sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} |\Delta_p^{\text{pri}}| \\
& - \sum_{p_0 \neq p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(\frac{1}{2} + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)}
\end{aligned}$$

where we have used (4.14) in (4.15). By definition, $|\tilde{S}_\alpha| \leq n + \ell_\alpha + g_\alpha$. Let

$$\epsilon_0 = \frac{\epsilon}{11(g_\alpha + \ell_\alpha + 1) + n} \leq \frac{\epsilon}{10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|},$$

by (4.13) we obtain

$$(4.16) \quad \frac{\epsilon_0}{w(\tilde{J}, p_0)} \leq \frac{\epsilon}{w(\tilde{J}, p_0)(10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|)} \leq \frac{\epsilon}{\deg X_\alpha}$$

If we let $M = M(\epsilon_0)$ be the constant fixed in Lemma 4.3 for ϵ_0 and choose

$$M' \geq (11(g_\alpha + \ell_\alpha + 1) + n)M \geq (10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|)M,$$

then $\deg X_\alpha \geq M'$ implies $w(\tilde{J}, p_0) > M$. In particular, we have $i_0(p_0) \in \mathbb{I}_\alpha^{\text{pri}}$. The whole term after (4.15) is equal to

$$\begin{aligned}
& = |\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\tilde{J}_\alpha} - \frac{\rho_{i_0(p_0)}}{2}|\{p_0\} \cap \tilde{S}_\alpha| + \sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} |\Delta_p^{\text{pri}}| \\
& - \sum_{\substack{p_0 \neq p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha^{\text{pri}}}} \frac{\rho_{i_0(p)}}{2} - \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)}.
\end{aligned}$$

Applying Lemma 4.2 to the term $|\Delta_{p_0}^{\text{pri}}| + 2(\deg X_\alpha - w_\alpha^{\text{pri}})\rho_{\tilde{J}_\alpha}$, Lemma 4.3 to the term $\sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} |\Delta_p^{\text{pri}}| - \sum_{p_0 \neq p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha, i_0(p) \in \mathbb{I}_\alpha^{\text{pri}}} \frac{\rho_{i_0(p)}}{2}$ in the above identity and using (4.16) we obtain

$$\begin{aligned}
& \frac{e_{X_\alpha}(\mathcal{J}(\lambda))}{2} - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\
& \leq \left(1 + \frac{\epsilon_0}{w(\tilde{J}, p_0)}\right) \left(\sum_{i \in \mathbb{I}_{p_0}^{\text{pri}}} \delta(\tilde{s}_i, p_0)\rho_i - \rho_{i_0(p_0)} \right) + \frac{\rho_{i_0(p_0)}}{2}(1 - |\{p_0\} \cap \tilde{S}_\alpha|) + \\
& + \left(\sum_{p_0 \neq p \in \tilde{\Lambda}_\alpha} \sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\rho_i - \sum_{\substack{p_0 \neq p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha^{\text{pri}}}} \rho_{i_0(p)} \right) - \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha \\ i_0(p) \in \mathbb{I}_\alpha \setminus \mathbb{I}_\alpha^{\text{pri}}}} \frac{\rho_{i_0(p)}}{2} \\
& \leq \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \left(\sum_{p \in \tilde{\Lambda}_\alpha} \sum_{i \in \mathbb{I}_p^{\text{pri}}} \delta(\tilde{s}_i, p)\rho_i - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \right) \\
& = \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \left(\sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \rho_{i_0(p)} \right) \\
& = \frac{E_\alpha(\rho)}{2} - \sum_{p \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2}.
\end{aligned}$$

This completes the proof the theorem. \square

5. STABILITY OF WEIGHTED POINTED NODAL CURVE

We prove Theorem 1.5 in this section. By Proposition 3.6, it suffices to verify the positivity of the \mathbf{a} - λ -weight $\omega_{\mathbf{a}}(\lambda)$ of $\text{Chow}(X, \mathbf{x}) \in \Xi$ for any staircase λ . Let \mathbf{s} be a diagonalizing basis of λ :

$$\lambda(t) := \text{diag}[t^{\rho_0}, \dots, t^{\rho_m}] \cdot t^{-\rho_{ave}}, \quad \text{with } \rho_0 \geq \rho_1 \geq \dots \geq \rho_m = 0.$$

The \mathbf{a} - λ -weight of $\text{Chow}(X, \mathbf{x})$ is the sum of the contributions from $\text{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ and $(\mathbb{P}W)^n$. By Proposition 2.1, the contribution from $\text{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is $\omega(\lambda)$.

For the contribution from $(\mathbb{P}W)^n$, we introduce subspaces

$$(5.1) \quad W_i = W_i(\lambda) := \{v \in W \mid s_i(v) = \dots = s_m(v) = 0\} \subset W = H^0(\mathcal{O}_X(1))^\vee.$$

They form a strictly increasing filtration of W . Also, for any closed subscheme $\Sigma \subset X$, we denote by

$$(5.2) \quad W_\Sigma := \{v \in W \mid s(v) = 0 \text{ for all } s \in H^0(\mathcal{O}_X(1) \otimes \mathcal{I}_\Sigma)\} \subset W$$

the *linear subspace spanned by* $\Sigma \subset X$. For instance, for a marked point x_i , W_{x_i} is the line in W spanned by $x_i \in \mathbb{P}W$; for any i and

$$(5.3) \quad Z_i = \{s_i = \dots = s_m = 0\} \subset X,$$

we let $W_i = W_{Z_i}$.

By [14, Prop 4.3], the \mathbf{a} - λ -weight of $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{P}W)^n$ is

$$(5.4) \quad \mu_{\mathbf{a}}(\lambda) := \sum_{j=1}^n a_j \left(\frac{\sum_{i=0}^m \rho_i}{m+1} + \sum_{i=0}^{m-1} (\rho_{i+1} - \rho_i) \dim(W_{x_j} \cap W_{i+1}(\lambda)) \right).$$

($\mu_{\mathbf{a}}(\lambda)$ implicitly depends on ρ_i , which we fix for the moment.) Therefore, the \mathbf{a} - λ -weight $\omega_{\mathbf{a}}(\lambda)$ of $\text{Chow}(X, \mathbf{x}) \in \Xi$ is

$$(5.5) \quad \omega_{\mathbf{a}}(\lambda) = \omega(\lambda) + \mu_{\mathbf{a}}(\lambda).$$

We now argue that for the staircase λ' constructed from λ by applying Proposition 3.6, we have

$$(5.6) \quad \omega_{\mathbf{a}}(\lambda) \geq \omega_{\mathbf{a}}(\lambda').$$

Indeed, since $\omega(\lambda) \geq \omega(\lambda')$, it suffices to show that $\mu_{\mathbf{a}}(\lambda) \geq \mu_{\mathbf{a}}(\lambda')$. To see that, we first notice that

$$(5.7) \quad \dim(W_{x_j} \cap W_{i+1}(\lambda)) = \#(x_i \cap \text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_{i+1})).$$

(Here $\mathcal{E}(\lambda)_i = (s_i, s_{i+1}, \dots, s_m) \subset \mathcal{O}_X(1)$.) On the other hand, by the proof of Proposition 3.6, we conclude

$$\text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_i) \subset \text{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda')_i).$$

This together with (5.7) proves

$$\dim(W_{x_j} \cap W_{i+1}(\lambda)) \leq \dim(W_{x_j} \cap W_{i+1}(\lambda')).$$

The inequality $\mu_{\mathbf{a}}(\lambda) \geq \mu_{\mathbf{a}}(\lambda')$ then follows from $\rho_i \geq \rho_{i+1}$. Therefore, to prove Theorem 1.5, we suffices to show $\omega_{\mathbf{a}}(\lambda) > 0$ for all staircase 1-PS's λ . From now on we assume λ is a *staircase*.

Before we proceed, we collect a few boundness results that are needed to pass from the estimates on single component in Section 4 to the entire curve.

Lemma 5.1. *Suppose $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope stable and $\chi_{\mathbf{a}}(X) > 0$ (cf. Theorem 1.5). Then there are positive constants M and C depending only on g, n and $\mathbf{a} \in \mathbb{Q}^n$ such that whenever $\deg X > M$ we have $\deg Y \geq C \deg X$ for any connected proper subcurve $Y \subset X$.*

In case $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope semi-stable, then the same conclusion $\deg Y \geq C \deg X$ holds except when Y is a line, $Y \cap \mathbf{x} = \emptyset$, $|Y \cap Y^{\mathbb{C}}| = 2$, and the inequality (1.6) is an equality.

Proof. We continue to denote $\ell_Y = |Y \cap Y^{\mathbb{C}}|$. Let g_Y be the arithmetic genus of Y . Suppose $2g_Y + \ell_Y \geq 3$, then since $a_i \geq 0$, the inequality (1.6) implies

$$\begin{aligned} \frac{\deg Y + \sum_{x_j \in Y} a_j/2}{g_Y - 1 + \sum_{x_j \in Y} a_j/2 + \ell_Y/2} &\geq \frac{\deg X + \sum_{j=1}^n a_j/2}{g - 1 + \sum_{j=1}^n a_j/2} - \frac{\ell_Y}{2(g_Y - 1) + \sum_{x_j \in Y} a_j + \ell_Y} \\ &\geq \frac{\deg X + \sum_{j=1}^n a_j/2}{g - 1 + \sum_{j=1}^n a_j/2} - 3. \end{aligned}$$

Denoting $\chi_{\mathbf{a}} = \chi_{\mathbf{a}}(X)$, this inequality implies

$$\deg Y \geq \left(\frac{\deg X}{2\chi_{\mathbf{a}}} - 6 \right) - \frac{n}{2}.$$

Therefore, if we choose $M' = 4\chi_{\mathbf{a}} \cdot (6 + n/2)$, choose $C' = 1/4\chi_{\mathbf{a}}$ and require $\deg X \geq M'$, we obtain

$$\deg Y \geq \deg X/4\chi_{\mathbf{a}} = C' \cdot \deg X.$$

Now suppose $2g_Y + \ell_Y \leq 2$. Since $Y \subset X$ is a proper connected subcurve, $\ell_Y \geq 1$ and $g_Y \geq 0$. Thus $g_Y = 0$ and $\ell_Y \leq 2$. In this case, (1.6) becomes

$$(5.8) \quad \left| \deg Y + \sum_{x_j \in Y} \frac{a_j}{2} - \frac{\deg X + \sum_{j=1}^n a_j/2}{g - 1 + \sum_{j=1}^n a_j/2} \cdot \left(-1 + \sum_{x_j \in Y} a_j/2 + \ell_Y/2 \right) \right| \leq 1.$$

Let $A := -1 + \sum_{x_j \in Y} a_j/2 + \ell_Y/2$. In case $A \leq 0$, then we have $\deg Y = 1$, $\sum_{x_i \in Y} a_i = 0$ and $\ell_Y = 2$. This is precisely the second case in the statement of the Lemma.

In case $A > 0$, then

$$\deg Y > \frac{\deg X + \sum_{j=1}^n a_j/2}{\chi_{\mathbf{a}}} \cdot 2A - 1 - \sum_{x_j \in Y} \frac{a_j}{2} > \frac{\deg X}{\chi_{\mathbf{a}}} A,$$

provided $\deg X > (2 + n)\chi_{\mathbf{a}}/2A$. To obtain a universal constant, we introduce

$$C_k := \min_{I \subset \{1, \dots, n\}} \left\{ \sum_{i \in I} a_i/2 + k/2 \mid \sum_{i \in I} a_i/2 + k/2 > 0 \right\},$$

and define $C_{\min} := \min_{k \geq 0} \{C_k\}$. Clearly, $C_{\min} > 0$. By our construction, $A \geq C_{\min}$ when $A > 0$. We choose $C'' = \min\{C_{\min}/\chi_{\mathbf{a}}, 1/2\chi_{\mathbf{a}}\}$, and choose

$$M'' := \max\{6g + n/2 - 6, \chi_{\mathbf{a}}(2 + n)/2C_{\min}\}.$$

Our discussion shows that the statement (1) in the Lemma holds in the case under study with this choices of M'' and C'' . Finally, we let $C = \min\{C', C''\}$ and $M = \max\{M', M''\}$. The Lemma holds with this choice of C and M in all cases. This completes the proof of the Lemma. \square

Corollary 5.2. *Let M be as in Lemma 5.1 and suppose $\deg X \geq M$. Then the number of irreducible components of X is at most $9g + 4n - 5$; the number of nodes of X is at most $10g + 4n - 5$.*

Proof. We divide the irreducible components of X into three categories: the first (resp. second; resp. third) consists of irreducible components of $X \in \mathbb{P}W$ that are not lines (resp. are lines that contains marked points; resp. are lines in $\mathbb{P}W$ that contains no marked points).

Applying the previous Lemma, for X_α in the first category, $\deg X_\alpha > \deg X/4\chi_{\mathbf{a}}$; thus the first category contains no more than $4\chi_{\mathbf{a}}$ elements. Since every component in the second category contains at least one marked point, there are at most n elements in this category.

We now bound the element in the third category. We let B_1 be a maximal subcollection of the third category so that $X - \cup_{\alpha \in B_1} X_\alpha$ remains connected. We let $Y = \overline{X - \cup_{\alpha \in B_1} X_\alpha}$. Then $g(Y) = g - |B_1| \geq 0$ since Y is connected. We let B_2 be the complement of B_1 in the third category. By Lemma 5.1, X_α and $X_{\alpha'}$ are disjoint for $\alpha \neq \alpha' \in B_2$. Thus if we let Y_1, \dots, Y_k be the connected components of $\overline{Y - \cup_{\alpha \in B_2} X_\alpha}$, then $|B_2| \leq k - 1$. On the other hand, applying Lemma 5.1 we know each Y_i has degree at least $\deg X/4\chi_{\mathbf{a}}$. Thus $k \leq 4\chi_{\mathbf{a}}$.

Combined, the total number r of irreducible components of X is bounded by

$$r \leq 4\chi_{\mathbf{a}} + n + (g + 4\chi_{\mathbf{a}} - 1) \leq 9g + 4n - 5.$$

Next we bound the total nodes of X . We first pick a maximal set $A_1 \subset X_{\text{node}}$ so that $X - A_1$ is connected; then $|A_1| \leq g$. We let $A_2 = X_{\text{node}} - A_1$. Then $|A_2|$ is less than the number of irreducible components of X . By the bound we just derived, we obtain $|X_{\text{node}}| \leq g + (9g + 4n - 5)$. This proves the Corollary. \square

We need the next consequence in our proof of Theorem 1.5 to replace inequality (1.6) by (1.4).

Corollary 5.3. *Let $X \subset \mathbb{P}H^0(\mathcal{O}_X(1))^\vee$ be a connected nodal curve of arithmetic genus $g > 0$. Suppose $\chi_{\mathbf{a}}(X) > 0$, then there is an M depending on g such that whenever $\deg X \geq M$, X satisfies (1.6) for any subcurve $Y \subset X$ if and only if it satisfies (1.4) for any subcurve $Y \subset X$.*

Proof. First, we claim that there is an M depending only on g, n such that for $\deg X \geq M$, we have $h^1(\mathcal{O}_X(1)|_Y) = 0$ for any subcurve $Y \subset X$ (not necessary proper) provided X satisfies either (1.6) or (1.4) for any proper subcurve $Y \subsetneq X$. To that, let $Y \subset X$ be any subcurve with arithmetic genus g_Y . Suppose $h^1(\mathcal{O}_X(1)|_Y) \geq 1$, then by the vanishing theorem we must have $g_Y > 1$ and $2 < \deg Y < 2g_Y - 1 \leq 2g - 1$.

Suppose X satisfies (1.6), then the claim is an easy consequence of Lemma 5.1 by letting $M = \max\{(2g + \nu)/C, 2g + \nu\}$ with C being chosen in Lemma 5.1 and ν being the total number of nodes of X . Suppose X satisfies (1.4), then the claim was proved in [7, Proposition 1.0.7].

Finally, when $h^1(\mathcal{O}_X(1)|_Y) = 0$ for any subcurve $Y \subset X$ the equivalence of (1.6) and (1.4) follows from an argument parallel to the one given in [2, Proposition 3.1] and [7, Proposition 1.0.7]. So we omit it. \square

Before we state our key estimate of this section, we first notice that λ being a staircase 1-PS (cf. Definition 3.8) implies that $\bigcup_{\alpha=1}^r \mathbb{I}_\alpha = \{0, \dots, m\}$, where \mathbb{I}_α is the index set of the component X_α defined in (3.3). We define the shifted weights $\{\hat{\rho}_i\}$ by

$$(5.9) \quad \hat{\rho}_i := \min_{\alpha} \{\rho_i - \rho_{h_\alpha} \mid i \in \mathbb{I}_\alpha\} \geq 0.$$

($\hat{\rho}_i$ may not be monotone. And $\hat{\rho}_i$ are only defined for staircase 1-PS.) We state out main estimate.

Proposition 5.4. *Suppose $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope stable (cf. (1.4)), and $\omega_X(\mathbf{a} \cdot \mathbf{x})$ is ample. Then for any $0 < \epsilon < 1$ there exists an M depending only on $\chi_{\mathbf{a}}(X)$ (cf. Theorem 1.5) and ϵ such that whenever $\deg X > M$, then for any staircase 1-PS λ we have*

$$(5.10) \quad \frac{E_X(\lambda, \rho)}{2} \leq \sum_{i=0}^m \rho_i - \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \sum_{\alpha=1}^r \left(\deg X_{\alpha} + \frac{\ell_{\alpha}}{2} - m_{\alpha} - 1 \right) \cdot \rho_{\tilde{h}_{\alpha}} + \frac{2C^{-1}\epsilon}{m+1} \sum_{i=0}^m \hat{\rho}_i.$$

where $\tilde{S}_{\text{reg}} := \bigcup_{\alpha=1}^r (\pi^{-1}(\mathbf{x}) \cap \tilde{X}_{\alpha} \cap \tilde{\Lambda})$ is the support of weighted points and $C > 0$ is the constant fixed in Lemma 5.1.

Proof. By the definition of $E_{\alpha}(\rho)$ (cf. (4.3)), $E_X(\lambda, \rho) = \sum_{\alpha=1}^r E_{\alpha}(\rho)$ is linear in $\rho = (\rho_i)$. By linear programming, (5.10) holds on

$$\mathbb{R}_+^{m+1} := \{(\rho_0, \dots, \rho_m) \in \mathbb{R}^{m+1} \mid \rho_0 \geq \rho_1 \geq \dots \geq \rho_m = 0\}$$

if and only if it holds on every edge of \mathbb{R}_+^{m+1} ; that is on

$$(5.11) \quad \rho = (\overbrace{1, \dots, 1}^{m_0}, 0, \dots, 0)$$

for every $0 < m_0 < m$.

We now fix a $0 < m_0 < m$. By possibly reindexing the irreducible components of X , we can assume that for an $\bar{r} \leq r$, $\tilde{h}_1 \leq \dots \leq \tilde{h}_{\bar{r}} < m_0 \leq \tilde{h}_{\bar{r}+1} \leq \dots \leq \tilde{h}_r$. In other words,

$$(5.12) \quad \rho_{\tilde{h}_1} = \dots = \rho_{\tilde{h}_{\bar{r}}} = 1, \quad \text{and} \quad \rho_{\tilde{h}_{\bar{r}+1}} = \dots = \rho_{\tilde{h}_r} = 0.$$

We let $Y := \bigcup_{\alpha > \bar{r}} X_{\alpha}$; let its complement $Y^{\complement} = \bigcup_{\alpha \leq \bar{r}} X_{\alpha}$.

We claim that Y^{\complement} is the maximal subcurve of X contained in the linear subspace $\mathbb{P}W_{m_0}$ (cf. (5.1)). By definition, for any α , \tilde{h}_{α} is the largest index $0 < i \leq m$ of which $s_i|_{X_{\alpha}} \neq 0$. On the other hand, because $\mathbb{P}W_{m_0} = \{s_{m_0} = \dots = s_m = 0\}$, $X_{\alpha} \subset \mathbb{P}W_{m_0}$ if and only if $s_i|_{X_{\alpha}} = 0$ for all $i \geq m_0$, which is equivalent to $\tilde{h}_{\alpha} < m_0$. This proves the claim.

Let X_{α} be a component in Y^{\complement} . Since $\rho_{\tilde{h}_{\alpha}} = 1$, $\rho_i = 1$ for $i \in \mathbb{I}_{\alpha}$. Using the explicit expression of $E_{\alpha}(\lambda, \rho)$, we obtain $E_{\alpha}(\lambda, \rho) = 2 \deg X_{\alpha}$. Thus

$$\sum_{\alpha \leq \bar{r}} E_{\alpha}(\lambda, \rho) = \sum_{\alpha \leq \bar{r}} 2 \deg X_{\alpha} = 2 \deg Y^{\complement}.$$

We now look at Y . Following (3.9) and (3.10), $\ell_Y := |Y \cap Y^{\complement}|$, and $\tilde{L}_Y := \pi^{-1}(Y \cap Y^{\complement}) \cap \tilde{Y}$. We claim that $\tilde{L}_Y \subset \tilde{\Lambda}_Y := \bigcup_{\alpha > \bar{r}} \tilde{\Lambda}_{\alpha}$. Indeed, for any $\alpha > \bar{r}$, there is an $i \geq m_0$ so that $s_i|_{X_{\alpha}} \neq 0$. However for any $\beta \leq \bar{r}$, $i \geq m_0$ implies $s_i|_{X_{\beta}} = 0$. Thus $s_i|_{X_{\alpha} \cap X_{\beta}} = 0$, and consequently, $\pi^{-1}(X_{\alpha} \cap X_{\beta}) \cap \tilde{X}_{\alpha} \subset \tilde{\Lambda}_{\alpha}$. Summing over all $\alpha > \bar{r}$ and $\beta \leq \bar{r}$, we obtain $\tilde{L}_Y \subset \tilde{\Lambda}_Y$. As a consequence,

$$(5.13) \quad \sum_{p \in \tilde{L}_Y} \rho_{i_0(p)} = \ell_Y.$$

To simplify the notation, in the remaining part of this section, we will abbreviate

$$\sum_{p \in A} \rho_{i_0(p)} := \sum_{p \in A \cap \tilde{\Lambda}} \rho_{i_0(p)},$$

with the understanding that for any $A \subset \tilde{X}$, the summation $\sum_{p \in A}$ only sums over $p \in A \cap \tilde{\Lambda}$.

Sublemma 5.5. *Let the notation be as before. Then*

$$\sum_{\alpha > \bar{r}} \frac{E_\alpha(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \leq \left(1 + \frac{\epsilon}{\deg X}\right) \left(\sum_{\substack{\alpha > \bar{r} \\ i \in \mathbb{I}_\alpha^{\text{pri}}}} \delta_\alpha(\tilde{s}_i) \rho_i - \sum_{p \in \tilde{L}_Y} \rho_{i_0(p)} - \sum_{p \in \tilde{N}_Y \setminus \tilde{L}_Y} \frac{\rho_{i_0(p)}}{2} \right) - \sum_{\pi(p) \in \mathbf{x} \cap Y} \frac{\rho_{i_0(p)}}{2}$$

Proof. We let $X_\alpha \subset Y$ be an irreducible component, then $\alpha > \bar{r}$ and $\rho_{h_\alpha} = 0$. Since $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope semi-stable, by Corollary 5.1, there are positive constants C and N such that whenever $\deg X \geq N$, either $\deg X_\alpha > C \deg X$ or $\deg X_\alpha = 1$. If $\deg X_\alpha > C \deg X$, from the definition of $E_\alpha(\rho)$ (cf. (4.3)) and $\rho_{h_\alpha} = 0$, we have

$$(5.14) \quad \frac{E_\alpha(\lambda, \rho)}{2} \leq \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i - \left(\frac{1}{2} + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)}.$$

If $\deg X_\alpha = 1$, (5.14) remain holds since by (4.4) and Definition 3.12, we have

$$\begin{aligned} \frac{E_\alpha(\lambda, \rho)}{2} &= \delta_\alpha(\tilde{s}_{\text{ind}_\alpha^{-1}(0)}) \cdot \frac{\bar{\rho}_{\text{ind}_\alpha^{-1}(0)}}{2} + \rho_{h_\alpha} = \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \cdot \frac{\rho_i}{2} \\ &\leq \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i - \left(\frac{1}{2} + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)}. \end{aligned}$$

Next we split

$$- \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)} = - \sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \rho_{i_0(p)} - \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \rho_{i_0(p)} - \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)}.$$

Then using $\rho_i \geq 0$, we get

$$\begin{aligned} - \left(\frac{1}{2} + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)} &\leq - \sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(\frac{1}{2} + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \rho_{i_0(p)} - \\ &\quad - \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)} + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} \\ &\leq - \sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \frac{\rho_{i_0(p)}}{2} - \\ &\quad - \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)} + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2}. \end{aligned}$$

Putting together, we obtain

$$\begin{aligned} \frac{E_\alpha(\lambda, \rho)}{2} &\leq \left(1 + \frac{C^{-1} \cdot \epsilon}{\deg X}\right) \left(\sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i) \rho_i - \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)} - \right. \\ &\quad \left. - \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \frac{\rho_{i_0(p)}}{2} \right) + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2}. \end{aligned}$$

Summing over α and applying (5.13) prove the Lemma. \square

The following inequality is crucial for the proof of the Proposition.

Lemma 5.6. *For $1 \leq k \leq m_0$, we have*

$$\sum_{\substack{\alpha > \bar{r} \\ i \in \mathbb{I}_\alpha^{\text{pri}} \cap [0, k)}} \delta_\alpha(\tilde{s}_i) \rho_i - \sum_{\substack{p \in \tilde{L}_Y \\ i_0(p) < k}} \rho_{i_0(p)} - \sum_{\substack{p \in \tilde{N}_Y \setminus \tilde{L}_Y \\ i_0(p) < k}} \frac{\rho_{i_0(p)}}{2} \leq \dim W_Y \cap W_k - \dim W_{Y \cap Y^{\mathbb{C}}} \cap W_k,$$

where $W_{Y \cap Y^{\mathbb{C}}}$ is the linear subspace in W spanned by $Y \cap Y^{\mathbb{C}}$.

Proof. We prove the Lemma by induction on k . When $k = 0$, then both sides of the inequality are zero, and the inequality follows. Suppose the Lemma holds for a $0 \leq k < m_0$. Then the Lemma holds for $k + 1$ if for the expressions

$$A_{k,1} := \sum_{\substack{\alpha > \bar{r}, \\ k \in \mathbb{I}_\alpha^{\text{pri}}}} \delta_\alpha(\tilde{s}_k) \rho_k, \quad A_{k,2} := \sum_{\substack{p \in \tilde{L}_Y, \\ i_0(p) = k}} \rho_{i_0(p)}, \quad A_{k,3} := \sum_{\substack{p \in \tilde{N}_Y \setminus \tilde{L}_Y, \\ i_0(p) = k}} \frac{\rho_{i_0(p)}}{2}$$

and

$$B_{k,1} := \dim W_Y \cap W_{k+1} - \dim W_Y \cap W_k, \quad B_{k,2} := \dim W_{Y \cap Y^{\mathbb{C}}} \cap W_{k+1} - \dim W_{Y \cap Y^{\mathbb{C}}} \cap W_k,$$

the following inequality holds

$$(5.15) \quad A_{k,1} - A_{k,2} - A_{k,3} \leq B_{k,1} - B_{k,2}.$$

To study the left hand side of (5.15), we introduce the set

$$(5.16) \quad R_k = \{p \in \tilde{Y} \mid k \in \mathbb{I}_p^{\text{pri}}\}.$$

By Proposition 3.11 and 3.13, R_k can take three possibilities according to

$$(5.17) \quad \sum_{\alpha > \bar{r}, k \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_k)$$

taking values 0, 1 or ≥ 2 . Notice that if $A_{k,1} = 0$, then $A_{k,1} - A_{k,2} - A_{k,3} \leq 0$. The Lemma holds trivially in this case since the right hand side of (5.15) is non-negative. So from now on, we will assume that $A_{k,1} \geq 1$, in particular, (5.17) is positive.

We first observe that since $\dim W_{k+1} - \dim W_k = 1$, both $B_{k,1}$ and $B_{k,2}$ can only take values 0 or 1. We now investigate the case when $B_{k,2} = 1$.

Claim 5.7. *Suppose (5.17) is positive and $B_{k,2} = 1$. Then there is a $p \in R_k$ (cf.(5.16)) such that $i_0(p) = k$ and*

$$(5.18) \quad q = \pi(p) \in Y \cap Y^{\mathbb{C}} \cap (\mathbb{P}W_{k+1} - \mathbb{P}W_k).$$

Proof. Suppose (5.17) is positive then there is a $p \in \text{inc}(\tilde{s}_k) \cap \tilde{X}_\alpha$ with $\alpha > \bar{r}$ and $k \in \mathbb{I}_\alpha^{\text{pri}}$. Let Z_k be the subscheme defined in (5.3) and $W_{Z_k+q} \supsetneq W_k$ be defined in (5.2). Then $W_{k+1} = W_{Z_k+q}$, since $\dim W_{k+1} = \dim W_k + 1$. Suppose $q = \pi(p) \notin Y \cap Y^{\mathbb{C}}$ and $k \in \mathbb{I}_\alpha^{\text{pri}}$ then by applying the argument parallel to Proposition 3.11 and 3.13, we deduce

$$(5.19) \quad W_{Z_k+q} + W_{Y \cap Y^{\mathbb{C}}} \supsetneq W_{Z_k} + W_{Y \cap Y^{\mathbb{C}}}.$$

On the other hand, $B_{k,2} = 1$ implies that

$$\begin{aligned} \dim(W_k + W_{Y \cap Y^{\mathbb{C}}}) &= \dim W_k + \dim W_{Y \cap Y^{\mathbb{C}}} - \dim W_k \cap W_{Y \cap Y^{\mathbb{C}}} \\ &= \dim W_{k+1} + \dim W_{Y \cap Y^{\mathbb{C}}} - \dim W_{k+1} \cap W_{Y \cap Y^{\mathbb{C}}} \\ &= \dim(W_{k+1} + W_{Y \cap Y^{\mathbb{C}}}), \end{aligned}$$

which means $W_k + W_{Y \cap Y^{\mathbb{C}}} = W_{k+1} + W_{Y \cap Y^{\mathbb{C}}}$ contradicting to (5.19). So we must have $q \in Y \cap Y^{\mathbb{C}}$.

By definition, $q \in \mathbb{P}W_{k+1}$ (cf. (5.1)) implies that $s_i(q) = 0$ for $i \geq k+1$; $q \notin \mathbb{P}W_k$ implies that not all $s_i(q)$, $k \leq i \leq m$, are zero. Combined, we have $s_k(q) \neq 0$. This implies $i_0(q) = k$. As an easy consequence, this shows that $B_{k,2} = 1$ forces $W_Y \cap W_{k+1} \neq W_Y \cap W_k$, and hence $B_{k,1} = 1$. In particular, the right hand side of (5.15) is non-negative. This proves the Claim. \square

We complete our proof of Lemma (5.6). When (5.17) takes value 1, then R_k consists of a single point, say $p \in \tilde{Y}$. In case $\pi(p) \in Y$ is a smooth point of X , $A_{k,1} = 1$ and $A_{k,2} = A_{k,3} = 0$. We claim that $B_{k,1} = 1$ and $B_{k,2} = 0$. Indeed, if $B_{k,1} = 0$, then $\mathbb{P}W_Y \cap \mathbb{P}W_{k+1} = \mathbb{P}W_Y \cap \mathbb{P}W_k$, which is the same as $Y \cap (s_k = \cdots = s_m = 0) = Y \cap (s_{k+1} = \cdots = s_m = 0)$ as subschemes of Y . But this contradicts to $\sum_{\alpha > \bar{r}, k \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_k) = 1$. Thus $B_{k,1} = 1$. On the other hand, if $B_{k,2} = 1$, then Claim 5.7 shows that $R_k \cap \tilde{Y}$ contains an element in \tilde{L}_Y , contracting to our assumption that $R_k = \{p\}$ lies over a smooth point of X .

In case $p \in \tilde{L}_Y$, then the previous paragraph shows that $A_{k,1} = B_{k,1} = 1$, $A_{k,3} = 0$. For the values of $A_{k,2}$ and $B_{k,2}$, when $i_0(p) = k$, then both $A_{k,2} = B_{k,2} = 1$; when $i_0(p) \neq k$, then both $A_{k,2} = B_{k,2} = 0$. Therefore, (5.15) holds.

The last case is when $p \in \tilde{N}_Y - \tilde{L}_Y$. In this case, since the point p' in $\tilde{Y} \cap \pi^{-1}(\pi(p))$ other than p is not contained in R_k , either $i_0(p) \neq k$ or $i_0(p) = i_0(p') = k$ and $k \notin \mathbb{I}_{p'}^{\text{pri}}$. In both cases, $A_{k,1} = B_{k,1} = 1$, and $B_{k,2} = 0$; the inequality (5.15) holds.

Lastly, when (5.17) is bigger than 1, by Proposition 3.11 and 3.13, either $R_k = \{p_-, p_+\}$ such that $\pi(p_-) = \pi(p_+)$ is a node of Y , i.e. $p_\pm \in \tilde{N}_Y$, and $i_0(p_-) = i_0(p_+) = k$, or $R_k = \{p_1, \dots, p_l\}$ so that $i_0(p_i) = k$ and $\{\pi(p_i)\}_{1 \leq i \leq l}$ are distinct nodes of X . In case $R_k = \{p_-, p_+\}$; since $p_\pm \in \tilde{N}_Y \setminus \tilde{L}_Y$, $A_{k,1} = 2$, $A_{k,2} = B_{k,2} = 0$, and $A_{k,3} = B_{k,1} = 1$. The inequality (5.15) holds in this case.

The other case is when $R_k = \{p_1, \dots, p_l\}$. By reindexing, we may assume p_1, \dots, p_{l_1} are in $\tilde{N}_Y \setminus \tilde{L}_Y$ and p_{l_1+1}, \dots, p_l are in \tilde{L}_Y . We let $p'_i \in \tilde{Y}$ be such that $\pi^{-1}(\pi(p_i)) = \{p_i, p'_i\}$ for $i \leq l_1$. Then $i_0(p'_i) = k$ as well, but $k \notin \mathbb{I}_{p'_i}^{\text{pri}}$ because of Proposition 3.11 and 3.13.

This in particular implies that the interior linking nodes $\tilde{N}_Y \setminus \tilde{L}_Y$ contributes once in $A_{k,1}$ but twice in $A_{k,3}$ (, e.g. only $\rho_{i_0(p_i)}$ appears in $A_{k,1}$, but both $\rho_{i_0(p_i)}$ and $\rho_{i_0(p'_i)}$ appear in $A_{k,3}$). Therefore, $A_{k,1} = l$; $A_{k,2} = l - l_1$, and $A_{k,3} = 2l_1/2 = l_1$. Hence the left hand side of (5.15) is 0. This proves (5.15) in this case; hence for all cases. This proves the Lemma. \square

We continue our proof of Proposition 5.4. We apply Lemma 5.5 and Lemma 5.6 with $k = m_0$. Noticing $\rho_{i_0(p)} = 0$ for $i_0(p) > m_0$, we obtain

$$\begin{aligned}
(5.20) \quad & \frac{E_Y(\lambda, \rho)}{2} - \frac{\ell_Y}{2} = \sum_{\alpha=\bar{r}+1}^r \frac{E_\alpha(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \leq \\
& \leq \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \left(\dim W_Y \cap W_{m_0} - \ell_Y\right) - \frac{1}{2} \sum_{\pi(p) \in \mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y} \rho_{i_0(p)} \cdot \\
& \leq \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \left(\dim W_Y \cap W_{m_0} - \ell_Y\right) - \frac{1}{2} \sum_{p \in \tilde{S}_{\text{reg}}} \hat{\rho}_{i_0(p)}.
\end{aligned}$$

Here we used that for all $p' \in \pi(\tilde{S}_{\text{reg}}) - X \cap \pi(\tilde{\Lambda}) \cap Y$, $\rho_{i_0}(p') = 0$. And the last inequality holds since by the Definition of \tilde{S}_{reg} and $\hat{\rho}_i$ (cf. (5.9)), we have $\sum_{q \in \tilde{S}_{\text{reg}}} \hat{\rho}_{i_0}(q) \leq \sum_{\pi(p) \in \mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y} \rho_{i_0}(p)$.

Using $E_{Y^{\mathfrak{C}}}(\lambda, \rho) = 2 \deg Y^{\mathfrak{C}}$ and $\deg X - g = m$, we obtain

$$\begin{aligned} \frac{E_X(\lambda, \rho)}{2} &= \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} \right) + \left(\frac{E_Y(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \right) \\ &\leq \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} \right) + \left(1 + \frac{2C^{-1}\epsilon}{m+1} \right) (m_0 + 1 - \dim W_Y) - \frac{1}{2} \sum_{p \in \tilde{S}_{\text{reg}}} \rho_{i_0}(p) , \end{aligned}$$

Here the last inequality follows from

$$\begin{aligned} \dim W_{m_0} &\geq \dim(W_{m_0} \cap W_Y + W_{m_0} \cap W_{Y^{\mathfrak{C}}}) \\ &= \dim W_{m_0} \cap W_Y + \dim W_{m_0} \cap W_{Y^{\mathfrak{C}}} - \dim W_{m_0} \cap W_Y \cap W_{Y^{\mathfrak{C}}} \\ &= \dim W_{m_0} \cap W_Y + \dim W_{Y^{\mathfrak{C}}} - \ell_Y . \end{aligned}$$

Now we consider the right hand side of (5.10) for ρ chosen as in (5.11), which gives $\sum_{i=0}^m \rho_i = m_0 + 1$. Since by our assumption, the embedding $X \subset \mathbb{P}W$ is given by a complete linear system of a very ample line bundle $\mathcal{O}_X(1)$, using our choice of weights ρ_i (cf. (5.12)),

$$\sum_{\alpha=1}^r \left(\deg X_{\alpha} + \frac{\ell_{\alpha}}{2} - m_{\alpha} - 1 \right) \cdot \rho_{h_{\alpha}} = \deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} - \dim W_{Y^{\mathfrak{C}}}.$$

We claim that $\sum_{i=0}^m \hat{\rho}_i = m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}$. Indeed, from our choice of ρ and the definition of $\hat{\rho}$ (cf. (5.9)), for any $0 \leq i \leq m$, $\hat{\rho}_i = 1$ or 0 , and it is 0 if and only if either $i > m_0$ or there is an X_{α} with $i \in \mathbb{I}_{\alpha}$ (cf. (3.3)) such that $\rho_{h_{\alpha}} = 1$, that is, $i \in \mathbb{I}_{Y^{\mathfrak{C}}} = \bigcup_{X_{\alpha} \subset Y^{\mathfrak{C}}} \mathbb{I}_{\alpha}$. This proves

$$\sum_{i=0}^m \hat{\rho}_i = m_0 + 1 - |\mathbb{I}_{Y^{\mathfrak{C}}}|.$$

Our claim will follow if once we prove $|\mathbb{I}_{Y^{\mathfrak{C}}}| = \dim W_{Y^{\mathfrak{C}}}$; but this follows from the criteria

$$(5.21) \quad i \in \mathbb{I}_{Y^{\mathfrak{C}}} \quad \text{if and only if} \quad \dim W_{i+1} \cap W_{X_{\alpha}} - \dim W_i \cap W_{X_{\alpha}} = 1 \text{ for some } X_{\alpha} \subset Y^{\mathfrak{C}}.$$

To justify this criteria, we notice that $\dim W_{i+1} \cap W_{X_{\alpha}} = \dim W_i \cap W_{X_{\alpha}}$ for all $X_{\alpha} \subset Y^{\mathfrak{C}}$ is equivalent to $Y^{\mathfrak{C}} \cap \{s_i = \dots = s_m = 0\} = Y^{\mathfrak{C}} \cap \{s_{i+1} = \dots = s_m = 0\}$ as subschemes of $Y^{\mathfrak{C}}$; that is, $\text{inc}(s_i) \cap Y^{\mathfrak{C}} = \emptyset$. Since λ is a staircase

$$i \notin \mathbb{I}_{\alpha} \text{ for all } X_{\alpha} \subset Y^{\mathfrak{C}} \quad \text{if and only if} \quad \text{inc}(s_i) \cap Y^{\mathfrak{C}} = \emptyset \text{ (cf. (3.3))} .$$

This proves (5.21).

With those in hand, we obtain

$$\begin{aligned}
\frac{E_X(\lambda, \rho)}{2} &= \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} \right) + \left(\frac{E_Y(\lambda, \rho)}{2} - \frac{\ell_Y}{2} \right) \\
&\leq \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} \right) + \left(1 + \frac{2C^{-1}\epsilon}{m+1} \right) (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}) - \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} \\
&\leq m_0 + 1 - \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} - \dim W_{Y^{\mathfrak{C}}} \right) + \frac{2C^{-1}\epsilon}{m+1} (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}) \\
&= \sum_{i=0}^m \rho_i - \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \sum_{\alpha=1}^r \left(\deg X_{\alpha} + \frac{\ell_{\alpha}}{2} - m_{\alpha} - 1 \right) \cdot \rho_{h_{\alpha}} + \frac{2C^{-1}\epsilon}{m+1} \sum_{i=0}^m \hat{\rho}_i.
\end{aligned}$$

So the proof of Proposition is completed. \square

We state and prove the main result of this section. We introduce

$$(5.22) \quad \hat{\omega}(\lambda, \rho) := \frac{2 \deg X}{m+1} \sum_{i=0}^m \rho_i - E_X(\lambda, \rho) \quad \text{and} \quad \hat{\omega}_{\mathbf{a}}(\lambda) = \hat{\omega}(\lambda) + \mu_{\mathbf{a}}(\lambda).$$

where $E_X(\lambda, \rho) := \sum_{\alpha=1}^r E_{\alpha}(\rho)$. By Theorem 4.1, we have $\omega(\lambda) \geq \hat{\omega}(\lambda)$.

Theorem 5.8. *Let $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ be a connected weighted pointed nodal curve that is slope stable. Suppose $\omega_X(\mathbf{a} \cdot \mathbf{x})$ is ample. We let $1 > \epsilon > 0$ be such that*

$$2(2C^{-1} + 1)\epsilon < \deg \omega_X(\mathbf{a} \cdot \mathbf{x})$$

with C given in Lemma 5.1. Then there exists an M depending only on $\chi_{\mathbf{a}}(X)$ and ϵ such that whenever $\deg X > M$, then for any staircase 1-PS λ we have

$$(5.23) \quad \omega_{\mathbf{a}}(\lambda) = \omega(\lambda) + \mu_{\mathbf{a}}(\lambda) \geq \hat{\omega}(\lambda) + \mu_{\mathbf{a}}(\lambda) \geq \frac{2 \cdot \epsilon}{m+1} \sum_{i=0}^m \hat{\rho}_i.$$

Proof. By Proposition 5.4, it suffices to prove

$$\begin{aligned}
(5.24) \quad \sum_{i=0}^m \rho_i - \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \sum_{\alpha=1}^r \left(\deg X_{\alpha} + \frac{\ell_{\alpha}}{2} - m_{\alpha} - 1 \right) \cdot \rho_{h_{\alpha}} + \frac{(2C^{-1} + 1)\epsilon}{m+1} \sum_{i=0}^m \hat{\rho}_i &\leq \\
&\leq \frac{\deg X}{m+1} \sum_{i=0}^m \rho_i + \frac{\mu_{\mathbf{a}}(\lambda, \rho)}{2}.
\end{aligned}$$

By linear programming, we only need to prove the above estimate for ρ of the form (5.11).

We will break the verification into several inequalities. First, we have

$$(5.25) \quad \mu_{\mathbf{a}}(\lambda, \rho) = \sum_{j=1}^n a_j \frac{m_0 + 1}{m+1} - \sum_{x_j \in Y \cap \mathbb{P}W_{m_0}} a_j - \sum_{x_j \in Y^{\mathfrak{C}} \cap \mathbb{P}W_{m_0}} a_j.$$

Here x_j runs through all marked points of the curve. We claim that

$$(5.26) \quad \sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} = \frac{|\mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y \cap \mathbb{P}W_{m_0}|}{2} \geq \sum_{x_j \in Y \cap \mathbb{P}W_{m_0}} \frac{a_j}{2}.$$

To this purpose, we first show that

$$(5.27) \quad \mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y \cap \mathbb{P}W_{m_0} = \mathbf{x} \cap Y \cap \mathbb{P}W_{m_0}.$$

Indeed, for any x_i in \mathbf{x} that lies in $Y \cap \mathbb{P}W_{m_0}$, $s_k(x_j) = 0$ for $k \geq m_0$. On the other hand, let $x_j \in X_\alpha \subset Y$; since $Y^{\mathfrak{C}}$ is the largest subcurve of X contained in $\mathbb{P}W_{m_0}$, for some $k \geq m_0$, $s_k|_{X_\alpha} \neq 0$. Combined with $s_k(x_j) = 0$, we conclude $x_j \in \pi(\tilde{\Lambda})$ (cf. Definition 3.1). In particular $\mathbf{x} \cap Y \cap \mathbb{P}W_{m_0} \subset \pi(\tilde{\Lambda})$. This proves (5.27).

Applying (5.27), and using that for any colliding subset $\{x_{i_1}, \dots, x_{i_s}\}$ (i.e. $x_{i_1} = \dots = x_{i_s}$) necessarily $a_{i_1} + \dots + a_{i_s} \leq 1$, we obtain

$$(5.28) \quad \sum_{x_j \in Y \cap \mathbb{P}W_{m_0}} \frac{a_j}{2} - \frac{|\mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y \cap \mathbb{P}W_{m_0}|}{2} = \sum_{x_j \in Y \cap \mathbb{P}W_{m_0}} \frac{a_j}{2} - \frac{|\mathbf{x} \cap Y \cap \mathbb{P}W_{m_0}|}{2} \leq 0,$$

hence (5.26).

By putting (5.25) and (5.26) together, we obtain

$$(5.29) \quad -\sum_{q \in \tilde{S}_{\text{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} - \frac{\mu_{\mathbf{a}}(\lambda, \rho)}{2} \leq -\frac{m_0 + 1}{m + 1} \sum_{j=1}^n \frac{a_j}{2} + \sum_{x_j \in Y^{\mathfrak{C}} \cap \mathbb{P}W_{m_0}} \frac{a_j}{2}.$$

On the other hand, for ρ of the form in (5.11), we have

$$(5.30) \quad \begin{aligned} & \sum_{i=0}^m \rho_i + \sum_{\alpha=1}^r \left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1 \right) \cdot \rho_{\bar{h}_\alpha} + \frac{(2C^{-1} + 1)\epsilon}{m + 1} \sum_{i=0}^m \hat{\rho}_i \\ &= m_0 + 1 + \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} - \dim W_{Y^{\mathfrak{C}}} \right) + \frac{(2C^{-1} + 1)\epsilon}{m + 1} (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}). \end{aligned}$$

Plugging (5.30) and (5.29) into (5.24), and using the slope condition

$$\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} + \sum_{x_j \in Y^{\mathfrak{C}} \cap \mathbb{P}W_{m_0}} \frac{a_j}{2} \leq \frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{m + 1} \dim W_{Y^{\mathfrak{C}}},$$

we obtain

$$\begin{aligned}
& -\frac{\mu_{\mathbf{a}}(\lambda, \rho)}{2} + \frac{E_X(\lambda, \rho)}{2} + \frac{((2C^{-1} + 1)\epsilon)}{m+1} \sum_{i=0}^m \hat{\rho}_i \\
& \leq m_0 + 1 + \left(\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} + \sum_{x_j \in Y^{\mathfrak{C}} \cap \mathbb{P}W_{m_0}} \frac{a_j}{2} - \dim W_{Y^{\mathfrak{C}}} \right) - \frac{m_0 + 1}{m+1} \sum_{i=1}^n \frac{a_i}{2} + \\
& \quad + \frac{(2C^{-1} + 1)\epsilon}{m+1} (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}) \\
& = \frac{\deg Y^{\mathfrak{C}} + \frac{\ell_Y}{2} + \sum_{x_j \in Y^{\mathfrak{C}} \cap \mathbb{P}W_{m_0}} \frac{a_j}{2}}{\dim W_{Y^{\mathfrak{C}}}} \dim W_{Y^{\mathfrak{C}}} - \frac{m_0 + 1}{m+1} \sum_{i=1}^n \frac{a_i}{2} + \\
& \quad + \left(1 + \frac{(2C^{-1} + 1)\epsilon}{m+1} \right) (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}) \\
& \leq \frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{m+1} \dim W_{Y^{\mathfrak{C}}} + \left(1 + \frac{(2C^{-1} + 1)\epsilon}{m+1} \right) (m_0 + 1 - \dim W_{Y^{\mathfrak{C}}}) - \frac{m_0 + 1}{m+1} \sum_{i=1}^n \frac{a_i}{2} \\
& \leq \frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{m+1} \left(\dim W_{Y^{\mathfrak{C}}} + m_0 + 1 - \dim W_{Y^{\mathfrak{C}}} \right) - \frac{m_0 + 1}{m+1} \sum_{i=1}^n \frac{a_i}{2} \\
& = \frac{\deg X}{m+1} \cdot (m_0 + 1) = \frac{\deg X}{m+1} \sum_{i=0}^m \rho_i,
\end{aligned}$$

where we have use the assumption $2(2C^{-1} + 1)\epsilon < \deg \omega_X(\mathbf{a} \cdot \mathbf{x})$ to conclude

$$\frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{m+1} > 1 + \frac{(2C^{-1} + 1)\epsilon}{m+1}$$

in the fourth inequality. This completes the proof. \square

Proof of Theorem 1.5. Since $\hat{\rho}_i \geq 0$, the sufficiency follows from Theorem 5.8. We now prove the other direction. Let $Y \subset X$ be any proper subcurve; let $W_Y \subset W$ be the linear subspace spanned by Y , and let $m_0 + 1 = \dim W_Y$. We choose a 1-PS $\lambda = \text{diag}[t^{\rho_0}, \dots, t^{\rho_m}] \cdot t^{-\rho_{\text{ave}}}$ such that the corresponding filtration $\{W_i\}_{i=0}^m$ satisfies $W_{m_0+1} = W_Y$; we choose the weights $\{\rho_i\}$ be as in (5.11). Then

$$\mu_{\mathbf{a}} = \sum_{j=1}^n a_j \left(\frac{m_0 + 1}{m+1} \right) - \sum_{x_j \in \mathbb{P}W_Y} a_j.$$

Thus by Corollary 2.7 (cf. [16, Prop 5.5]), $e(\tilde{\mathcal{J}})/2 = \deg Y + \ell_Y/2$; hence

$$\begin{aligned}
0 \leq \frac{\hat{\omega} + \mu_{\mathbf{a}}}{2} &= \frac{m_0 + 1}{m+1} \cdot \deg X - \left(\deg Y + \frac{\ell_Y}{2} \right) + \frac{m_0 + 1}{m+1} \sum_{j=1}^n \frac{a_j}{2} - \sum_{x_j \in \mathbb{P}W_Y} \frac{a_j}{2} \\
&= (m_0 + 1) \left(\frac{\deg X + \sum_{j=1}^n \frac{a_j}{2}}{m+1} - \frac{\deg Y + \frac{\ell_Y}{2} + \sum_{x_j \in Y} \frac{a_j}{2}}{m_0 + 1} \right),
\end{aligned}$$

which is (1.4). This completes the proof of the Theorem. \square

6. RE-CONSTRUCTION OF THE MODULI OF WEIGHTED POINTED CURVES

In this section, we use GIT quotient of Hilbert scheme to construct the moduli of weighted pointed stable curves, first introduced and constructed by Hassett [9] using different method.

Definition 6.1. *A weighted pointed semi-stable curve is a weighted pointed curve $(X, \mathbf{x}, \mathbf{a})$ such that*

- (1) $\omega_X(\mathbf{a} \cdot \mathbf{x})$ is numerically non-negative;
- (2) the total degree $2\chi_{\mathbf{a}}(X) = \deg \omega_X(\mathbf{a} \cdot \mathbf{x})$ is positive;
- (3) for any smooth subcurve $E \subset X$ such that $\deg \omega_X(\mathbf{a} \cdot \mathbf{x})|_E = 0$, necessarily $E \cap \mathbf{x} = \emptyset$ and $E \cong \mathbb{P}^1$.

We call $E \subset X$ satisfying (3) exceptional components. We say $(X, \mathbf{x}, \mathbf{a})$ is weighted pointed stable if it does not contain exceptional components.

We fix integers n, g and weights $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a}}(X) > 0$; for a large integer k such that $k \cdot a_i \in \mathbb{Z}$ for all i , we let $d = (|\mathbf{a}| + 2g - 2) \cdot k$, and form

$$(6.1) \quad P(t) = d \cdot t + 1 - g \in \mathbb{Z}[t], \quad \text{and set } m = P(1).$$

We denote by $\text{Hilb}_{\mathbb{P}^m}^P$ the Hilbert scheme of subschemes of \mathbb{P}^m of Hilbert polynomial P ; we define \mathcal{H} be the fine moduli scheme of families of data

$$\mathcal{H} =_{\text{set}} \{(X, \iota, \mathbf{x}) \mid [\iota : X \rightarrow \mathbb{P}^m] \in \text{Hilb}_{\mathbb{P}^m}^P, \mathbf{x} = (x_1, \dots, x_n) \in X^n\}.$$

Using that Hilbert schemes are projective, we see that \mathcal{H} exists and is projective. We denote by

$$(6.2) \quad (\pi_{\mathcal{H}}, \varphi) : \mathcal{X} \longrightarrow \mathcal{H} \times \mathbb{P}^m, \quad \mathbf{x}_i : \mathcal{H} \rightarrow \mathcal{X}$$

the universal family of \mathcal{H} .

We introduce a parallel space for the Chow variety. We let $\text{Chow}_{\mathbb{P}^m}^d$ be the Chow variety of degree d dimension one effective cycles in \mathbb{P}^m . For any such cycle Z , we denote by $\text{Chow}(Z) \in \text{Div}^{d,d}[(\mathbb{P}^m)^\vee \times (\mathbb{P}^m)^\vee]$ its associated Chow point (cf. Section 1). We define

$$\mathcal{C} := \{(Z, \mathbf{x}) \in \text{Chow}_{\mathbb{P}^m}^d \times (\mathbb{P}^m)^n \mid \mathbf{x} = (x_1, \dots, x_n) \in \text{supp}(Z)^n\}.$$

By Chow Theorem, \mathcal{C} is projective. Using the Chow coordinate, we obtain an injective morphism

$$(6.3) \quad \mathcal{C} \xrightarrow{\subset} \text{Div}^{d,d}[(\mathbb{P}^m)^\vee \times (\mathbb{P}^m)^\vee] \times (\mathbb{P}^m)^n.$$

Like before (cf. Section 1), we endow it with an ample \mathbb{Q} -line bundle $\mathcal{O}_{\mathcal{C}}(1, \mathbf{a})$ (depending on the weights \mathbf{a}); the line bundle is canonically linearized by the diagonal action of

$$G := SL(m+1)$$

on (6.3). We let $\mathcal{C}^{ss} \subset \mathcal{C}$ be the (open) set of *semi-stable* points with respect to the G linearization on $\mathcal{O}_{\mathcal{C}}(1, \mathbf{a})$.

For any one-dimensional subscheme $X \subset \mathbb{P}^m$, we denote by $[X]$ its associated one-dimensional cycle. By sending $(X, \iota, \mathbf{x}) \in \mathcal{H}$ to $([X], \mathbf{x}) \in \mathcal{C}$, we obtain the G -linear *Hilbert-Chow* morphism (cf. [14, Section 5.4])

$$\Phi : \mathcal{H} \longrightarrow \mathcal{C}.$$

Lemma 6.2. *For fixed g, n and \mathbf{a} satisfying $\chi_{\mathbf{a}}(X) > 0$, there is an integer M depending on g, n and \mathbf{a} so that for $d \geq M$, $\Phi^{-1}(\mathcal{C}^{ss})$ consists exactly of those $(X, \iota, \mathbf{x}) \in \mathcal{H}$ so that the associated data $(X, \iota^* \mathcal{O}_{\mathbb{P}^m}(1), \mathbf{x}, \mathbf{a})$ is a slope semi-stable weighted nodal curve.*

Proof. By an argument parallel to [16, Prop. 5.5], one proves that there is an M depending only on $\chi_{\mathbf{a}}(X)$ such that for $d \geq M$, $\text{Chow}(X, \mathbf{x}) \in \mathcal{C}^{ss}$ implies that X is a nodal curve and the inclusion $\iota : X \rightarrow \mathbb{P}^m$ is given by a complete linear system.

We now show that any $(X, \mathbf{x}, \mathbf{a}) \in \Phi^{-1}(\mathcal{C}^{ss})$ is a weighted pointed nodal curve as defined in the beginning of the paper. We first check that the weighted points are away from the nodes, and the total weight at any point is no more than one.

Let $p \in X$; we choose the 1-PS λ as in the Example 2.8; the λ -weight for $\text{Chow}(X, \mathbf{x})$ is (cf.(5.5))

$$\omega(\lambda) + \mu_{\mathbf{a}}(\lambda) = \frac{2 \deg X}{m+1} - \epsilon_p + \frac{1}{m+1} \sum_{j=1}^n a_j - \sum_{x_j=p} a_j = 2 - \epsilon_p + \frac{2\chi_{\mathbf{a}}(X)}{m+1} - \sum_{x_j=p} a_j,$$

where $\epsilon_p = 2$ if p is a node and 1 otherwise. Since $\text{Chow}(X, \mathbf{x})$ is semistable, we must have $0 \leq \omega(\lambda) + \mu_{\mathbf{a}}(\lambda)$. Now we choose M' so that $m+1 = M' + 1 - g(X) > \frac{2\chi_{\mathbf{a}}(X)}{\min_{a_j > 0} \{a_i\}}$; then $0 \leq \omega(\lambda) + \mu_{\mathbf{a}}(\lambda)$ implies that the weighted points are away from the nodes, and the total weight of marked points at p does not exceed one.

Finally, the slope semi-stability of $(X, \mathbf{x}, \mathbf{a})$ follows from the necessity part of Theorem 1.5; the condition (3) of Definition 6.1 follows from the inequality (1.6) and Lemma 5.1. This proves that all $(X, \mathbf{x}, \mathbf{a}) \in \Phi^{-1}(\mathcal{C}^{ss})$ are slope-semi-stable weighted nodal curves. The other direction is straightforward, and will be omitted. \square

We define

$$\mathcal{H}^{ss} = \Phi^{-1}(\mathcal{C}^{ss}) \subset \mathcal{H}.$$

Corollary 6.3. *For $d \geq M$ specified in Lemma 6.2, the restriction*

$$\Phi^{ss} := \Phi|_{\mathcal{H}^{ss}} : \mathcal{H}^{ss} \rightarrow \mathcal{C}^{ss}$$

is an isomorphism

Proof. First, since \mathcal{H} is proper, Φ^{ss} is surjective. For the injectivity of Φ^{ss} , suppose there are (X, ι, \mathbf{x}) and $(X', \iota', \mathbf{x}') \in \mathcal{H}^{ss}$ such that $\Phi(X, \iota, \mathbf{x}) = \Phi(X', \iota', \mathbf{x}') \in \mathcal{C}^{ss}$, then by Lemma 6.2, both X and X' are nodal subcurves of \mathbb{P}^m . Since $\Phi(X, \iota, \mathbf{x}) = \Phi(X', \iota', \mathbf{x}') \in \mathcal{C}^{ss}$, the cycles $[X] = [X']$ and $\mathbf{x} = \mathbf{x}' \subset \mathbb{P}^m$; since both X and X' are nodal, we have $X = X'$. This proves $(X, \iota, \mathbf{x}) = (X', \iota', \mathbf{x}')$; thus Φ^{ss} is bijective. Finally, Φ^{ss} is an isomorphism since both \mathcal{H}^{ss} and \mathcal{C}^{ss} are smooth. \square

To construct the moduli of weighted pointed curves, taking the k specified before (6.1), we form

$$\mathcal{K} = \{(X, \iota, \mathbf{x}) \in \mathcal{H} \mid X \text{ smooth weighted pointed curves, } \iota^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}\}.$$

It is locally closed, and is smooth. Since X in $(X, \iota, \mathbf{x}) \in \mathcal{K}$ are smooth, applying Theorem 1.5, we conclude that $\Phi(\mathcal{K}) \subset \mathcal{C}^{ss}$, thus $\mathcal{K} \subset \mathcal{H}^{ss}$. Let $\overline{\mathcal{K}} \subset \mathcal{H}^{ss}$ be the closure of \mathcal{K} in \mathcal{H}^{ss} . Because Φ^{ss} is an isomorphism, and \mathcal{C} is projective, the GIT quotient $\mathcal{H}^{ss} // G \cong \mathcal{C}^{ss} // G$ exists and is projective. Because $\overline{\mathcal{K}}$ is closed in \mathcal{H}^{ss} , the GIT quotient

$$(6.4) \quad \mathbf{q} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{K}} // G$$

exists and is projective.

Theorem 6.4. *The coarse moduli space $\overline{\mathcal{M}}_{g, \mathbf{a}}$ of stable genus g , \mathbf{a} -weighted nodal curves constructed by Hassett is canonically isomorphic to the GIT quotient $\overline{\mathcal{K}} // G$.*

The main technical part of the proof is to analyze the closed points of $\overline{\mathcal{K}}//G$. We have the following preliminary results.

For any $(X, \iota, \mathbf{x}) \in \overline{\mathcal{K}}$, since the associated weighted pointed curve $(X, \mathbf{x}, \mathbf{a})$ is semistable, we can form a new weighted pointed curve by contracting all of its exceptional components (cf. Definition 6.1). We denote the resulting curve by

$$(6.5) \quad (X^{\text{st}}, \mathbf{x}^{\text{st}}, \mathbf{a}),$$

and call it the *stabilization* of $(X, \mathbf{x}, \mathbf{a})$. Since the marked points never lie on the contracted components, the stabilization produces a weighted pointed nodal curve of the same genus. Further, the stabilization also applies to families of semistable weighted curves. Thus applying this to the restriction to $\overline{\mathcal{K}}$ of the universal family of \mathcal{H} , we obtain a family of weighted pointed stable curves on $\overline{\mathcal{K}}$. Since $\overline{\mathcal{M}}_{g,\mathbf{a}}$ is the coarse moduli space of stable weighted pointed nodal curve, we obtain a morphism

$$(6.6) \quad \overline{\Psi} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{M}}_{g,\mathbf{a}}.$$

As this morphism is G -equivariant with G acting trivially on $\overline{\mathcal{M}}_{g,\mathbf{a}}$, it descends to a morphism

$$(6.7) \quad \psi : \overline{\mathcal{K}}//G \longrightarrow \overline{\mathcal{M}}_{g,\mathbf{a}}.$$

We will prove Theorem 6.4 by proving that ψ is an isomorphism.

6.1. Surjectivity. Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed stable curve. We endow it the polarization $\mathcal{O}_X(1) = \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}$ together with the embedding $\iota : X \rightarrow \mathbb{P}H^0(\mathcal{O}_X(1))^\vee$. When X is smooth, $(X, \iota, \mathbf{x}, \mathbf{a})$ lies in \mathcal{K} ; when X is singular, this may not necessarily hold. Our solution is to replace $\omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}$ by its twist, to be defined momentarily.

Given (X, \mathbf{x}) , we choose a smoothing $\pi : \mathcal{X} \rightarrow T$ over a pointed curve $0 \in T$ such that \mathcal{X} is smooth and $\mathcal{X}_0 = \mathcal{X} \times_T 0 \cong X$. By shrinking T if necessary, we can extend the n -marked points of X to sections $\mathbf{r}_i : T \rightarrow \mathcal{X}$ so that, denoting $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$, $(\mathcal{X}, \mathbf{r}, \mathbf{a})$ form a flat family of weighted pointed stable curves. Let X_1, \dots, X_r be the irreducible components of X . The following Proposition gives the surjectivity of ψ .

Proposition 6.5. *Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed stable curve, and let $(\mathcal{X}, \mathbf{r}, \mathbf{a})$ be the T -family constructed. Then there exist non-negative integers $\{b_\alpha\}_{\alpha=1}^r$ so that after letting*

$$(6.8) \quad \mathcal{O}_{\mathcal{X}}(1) = \omega_{\mathcal{X}/T}(\mathbf{a} \cdot \mathbf{s})^{\otimes k} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(\sum b_\alpha X_\alpha),$$

$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1), \mathbf{s})$ forms a family of slope semistable weighted pointed nodal curves.

The Proposition was essentially proved by Caporaso in [2]. Since we need to use the same technique to prove the injectivity, we recall the notation used to prove this Proposition. The remainder part of this subsection essentially follows [2].

For any line bundle \mathcal{L} on X , we denote $\delta_\alpha(\mathcal{L}) = \deg \mathcal{L}|_{X_\alpha}$. We define the associated lattice point of \mathcal{L} be

$$\vec{\delta}(\mathcal{L}) := (\delta_1(\mathcal{L}), \dots, \delta_r(\mathcal{L})) \in \mathbb{Z}^{\oplus r}.$$

We call $\vec{\delta}(\mathcal{L})$ the numerical class of \mathcal{L} .

We next introduce a subgroup $\Gamma_X \subset \mathbb{Z}^{\oplus r}$. We let

$$(6.9) \quad \ell_{\alpha,\beta} = \ell_{\alpha,\beta}(X) = |X_\alpha \cap X_\beta| \quad \text{if } \alpha \neq \beta; \quad \text{and} \quad \ell_{\alpha,\alpha} = \ell_{\alpha,\alpha}(X) = -|X_\alpha \cap X_\alpha^{\mathbb{C}}|.$$

We define $\vec{\ell}_\alpha = \vec{\ell}_\alpha(X) = (\ell_{\alpha,1}(X), \ell_{\alpha,2}(X), \dots, \ell_{\alpha,r}(X))$. We define $\Gamma_X \subset \mathbb{Z}^{\oplus r}$ be the subgroup generated by $\vec{\ell}_1, \dots, \vec{\ell}_r$.

Remark 6.6. Let $\mathcal{L} = \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}$. Since \mathcal{X} is smooth, for the invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ stated in (6.8) depending on the integers b_1, \dots, b_r , we have

$$\vec{\delta}(\mathcal{O}_{\mathcal{X}}(1)|_X) = \vec{\delta}(\mathcal{L}) + \sum_{\alpha=1}^r b_{\alpha} \vec{\ell}_{\alpha}.$$

This says that any two choices of $\mathcal{O}_{\mathcal{X}}(1)$ restricted to the central fiber have equivalent numerical classes modulo Γ_X . This motivates the definition

Definition 6.7. We define the degree class group of X be the quotient $\mathbb{Z}^{\oplus r}/\Gamma_X$.

We introduce one more notation. For any vector $\vec{v} = (v_1, \dots, v_r) \in \mathbb{Z}^{\oplus r}$ and any subcurve $Y \subset X$, mimicing the notion of degree, we define

$$\deg_Y \vec{v} = \sum_{X_{\alpha} \subset Y} v_{\alpha}.$$

In particular, $\deg_X \vec{v} = \sum v_{\alpha}$.

Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed nodal curve and \mathcal{L} a line bundle on X of total degree d . For any subcurve $Y \subset X$, we introduce the *extremes* of Y (depending on d) be

$$(6.10) \quad \mathbf{M}_Y^{\pm} := \frac{\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x})}{\deg_X \omega_X(\mathbf{a} \cdot \mathbf{x})} \left(d + \sum_{j=1}^n \frac{a_j}{2} \right) - \sum_{x_j \in Y} \frac{a_j}{2} \pm \frac{\ell_Y}{2}.$$

Proposition 1.6 is reformulated as

Lemma 6.8. Suppose $\chi_{\mathbf{a}}(X) > 0$, and let $d > M$, where M is defined in Corollary 5.3. Then a weighted pointed nodal curve $(X, \mathbf{x}, \mathbf{a})$ with a numerical effective line bundle \mathcal{L} on X of $\deg \mathcal{L} = d$ is slope semi-stable (cf. Proposition 1.6) if and only if

$$(6.11) \quad \deg_Y \mathcal{L} \in [\mathbf{M}_Y^-, \mathbf{M}_Y^+] \text{ for any subcurve } Y \subset X.$$

We quote the basic properties of extremes.

Lemma 6.9. Let Y, Y_1 and Y_2 be surcurves of X . We have

- (1) $\mathbf{M}_Y^+ - \mathbf{M}_Y^- = \ell_Y$, and $\mathbf{M}_{Y \cap Y_1}^- + \mathbf{M}_Y^+ = v_{Y \cap Y_1} + v_Y = d$;
- (2) suppose $E \subset X$ is an exceptional component such that $|E \cap Y| = 1$, then $\mathbf{M}_{E \cup Y}^{\pm} = \mathbf{M}_Y^{\pm}$;
- (3) suppose Y_1 and Y_2 have no common component, then $\mathbf{M}_{Y_1 \cup Y_2}^{\pm} \pm |Y_1 \cap Y_2| = \mathbf{M}_{Y_1}^{\pm} + \mathbf{M}_{Y_2}^{\pm}$.

Proof. The proof is a direct check, and will be omitted. \square

Let $\mathbb{Z}_{\geq 0}^{\oplus r}$ be those $\vec{v} = (v_i) \in \mathbb{Z}^{\oplus r}$ so that $v_i \geq 0$. We define

$$\mathfrak{B}_{X, \mathbf{x}, \mathbf{a}}^d = \{ \vec{v} \in \mathbb{Z}_{\geq 0}^{\oplus r} \mid \deg_X \vec{v} = d, \vec{v} \text{ satisfies (6.11) with } \deg_Y \mathcal{L} \text{ replaced by } \deg_Y \vec{v} \}.$$

Proposition 6.10. Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed semi-stable (cf. Definition 6.1) curve and d sufficiently large. Then for any $\vec{v} \in \mathbb{Z}^{\oplus r}$ satisfying $\deg_X \vec{v} = d$, we have

$$(\vec{v} + \Gamma_X) \cap \mathfrak{B}_{X, \mathbf{x}, \mathbf{a}}^d \neq \emptyset.$$

Proof. The proof is parallel to that of [2, Prop. 4.1], and will be omitted. \square

Lemma 6.11. Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed nodal curve satisfying $\chi_{\mathbf{a}}(X) > 0$, then there is a constant K depends only on the genus g , $\chi_{\mathbf{a}}(X)$ and \mathbf{a} such that if $d \geq K$, then $\mathbf{M}_Y^- > 0$ for any connected subcurve $Y \subset X$ satisfying $\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) > 0$.

Proof. Let $Y \subset X$ be a connected subcurve such that $\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) > 0$. Since the expression \mathbf{M}_Y^- only involves the nodes $L_Y = Y \cap Y^{\mathbb{C}}$, without loss of generality we can assume that X consists of two smooth irreducible components Y and $Y^{\mathbb{C}}$. Then $\ell_Y \leq g+1$. Thus to prove that for large d we have $\mathbf{M}_Y^- > 0$ whenever $\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) > 0$, it suffices to show that

$$(6.12) \quad \inf\{\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) \mid \deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) > 0\}$$

is bounded below by a positive constant depending only on \mathbf{a} and g . But this is true because (6.12) is bounded below by

$$\kappa = \inf\left\{\sum_{i \in I} a_i - l \mid l \in \mathbb{Z}, I \subset \{1, \dots, n\}, \sum_{i \in I} a_i - l > 0\right\}.$$

Since \mathbf{a} is fixed, κ is positive. Since $d = k \cdot \deg \omega_X(\mathbf{a} \cdot \mathbf{x})$, for large k , which is the same as for large d , we have $\mathbf{M}_Y^- > 0$ whenever $\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x}) > 0$. This proves the Lemma. \square

Proof of Proposition 6.5. By Proposition 6.10, there are $\{b_\alpha\}$'s such that for the $\mathcal{O}_{\mathcal{X}}(1)$ given in (6.8) and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X}_0}$, $\tilde{\delta}(\mathcal{L})$ satisfies (6.11). To show that $(X, \mathcal{L}, \mathbf{x}, \mathbf{a})$ is a polarized slope semi-stable curve, we need to show that \mathcal{L} is ample. Since $\tilde{\delta}(\mathcal{L})$ satisfies (6.11), \mathcal{L} is ample if $\mathbf{M}_{X_\alpha}^- > 0$ for any component $X_\alpha \subset X$; but this is precisely Lemma 6.11 because $(X, \mathbf{x}, \mathbf{a})$ is weighted pointed implies that $\omega_X(\mathbf{a} \cdot \mathbf{x})$ is positive. \square

6.2. Injectivity. In this subsection, we use the separatedness of $\overline{\mathcal{K}}//G$ to prove that ψ in (6.7) is injective.

Definition 6.12. Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed nodal curve (cf. Definition 1.1). We say a weighted pointed nodal curve $(\bar{X}, \bar{\mathbf{x}}, \mathbf{a})$ is a blow-up of $(X, \mathbf{x}, \mathbf{a})$ if there is a morphism $\pi : \bar{X} \rightarrow X$ that is derived by contracting some of the exceptional components of $(\bar{X}, \bar{\mathbf{x}}, \mathbf{a})$.

(Recall that exceptional component is defined in Definition (6.1).) Suppose $(\bar{X}, \bar{\mathbf{x}}, \mathbf{a})$ is a blow-up of $(X, \mathbf{x}, \mathbf{a})$, then $(\bar{X}^{\text{st}}, \bar{\mathbf{x}}^{\text{st}}, \mathbf{a}) = (X^{\text{st}}, \mathbf{x}^{\text{st}}, \mathbf{a})$ (cf. (6.5)).

Since the restriction of ψ to $\mathcal{K}//G$ is an isomorphism, ψ is a birational morphism. By Zariski's Main theorem and the properness of $\overline{\mathcal{K}}//G$, the injectivity follows from

Lemma 6.13. $\psi^{-1}(\psi(\xi))$ is zero dimensional for each $\xi \in \overline{\mathcal{K}}//G$.

Proof. Let $\xi \in \overline{\mathcal{K}}//G \setminus (\mathcal{K}//G)$, and let $\psi(\xi) = (X, \mathbf{x}, \mathbf{a}) \in \overline{\mathcal{M}}_{g, \mathbf{a}}$ be the associated weighted pointed stable curve. We describe the set $\Theta_\xi = \mathbf{q}^{-1}(\psi^{-1}(\psi(\xi))) \subset \overline{\mathcal{K}}$, where \mathbf{q} is defined in (6.4).

For any $\eta = (\bar{X}, \iota, \bar{\mathbf{x}}) \in \Theta_\xi \subset \overline{\mathcal{K}}$, there is a smooth affine curve $\phi : 0 \in T \rightarrow \overline{\mathcal{K}}$ so that the pull back of the universal family of $\overline{\mathcal{K}}$, say $\pi : (\mathcal{X}, \mathcal{L}, \mathbf{s}) \rightarrow T$, contains $(\bar{X}, \iota^* \mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}})$ as its central fiber and $\phi(T \setminus \{0\}) \subset \mathcal{K}$, and that the total space \mathcal{X} is smooth.

By Lemma 6.2, the central fiber $(\bar{X}, \bar{\mathbf{x}}, \mathbf{a})$ is weighted pointed semi-stable (cf. Definition 6.1) and is a blow-up of $(X, \mathbf{x}, \mathbf{a})$ (cf. Definition 6.12). Since \mathcal{X} is smooth, there are integers $\{b_\alpha\}$ indexed by the irreducible components \bar{X}_α of \bar{X} , such that if we view \bar{X}_α as divisor in \mathcal{X} , then

$$\iota^* \mathcal{O}_{\mathbb{P}^m}(1) = \omega_{\bar{X}/T}(\mathbf{a} \cdot \mathbf{x})^{\otimes k} (\sum_{\alpha=1}^{\bar{r}} b_\alpha \bar{X}_\alpha).$$

Since the collection of blow-ups of X coupled with integers $\{b_\alpha\}_{\alpha=1}^{\bar{r}}$ is a discrete set, the choices of $(\bar{X}, \mathcal{L}, \bar{\mathbf{x}})$ are discrete. Thus $\{(\bar{X}, \iota^* \mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}}) \mid (\bar{X}, \iota, \bar{\mathbf{x}}) \in \Theta_\xi\}$ is discrete. Finally, any two $(\bar{X}, \iota, \bar{\mathbf{x}})$ with isomorphic $(\bar{X}, \iota^* \mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}})$ lie in the same G -orbit. Thus Θ_ξ consists of a discrete collection of G -orbits. Hence $\psi^{-1}(\psi(\xi))$ is discrete. \square

We remark that this proof uses the existence of the coarse moduli space $\overline{\mathcal{M}}_{g,\mathbf{a}}$ constructed by Hassett.

6.3. The coarse moduli space. We prove that $\overline{\mathcal{K}}//G$ is a coarse moduli space of weighted pointed stable curves, thus proving that ψ is an isomorphism.

Proposition 6.14. *Let T be any scheme and $(\mathcal{X}, \mathbf{r}, \mathbf{a})$ be a T -family of weighted pointed stable curves. Then there is a unique morphism $f : T \rightarrow \overline{\mathcal{K}}//G$, canonical under base changes, such that for any closed point $c \in T$, the image $\psi(f(c)) \in \overline{\mathcal{M}}_{g,\mathbf{a}}$ is the closed point associated to the weighted pointed stable curve $(\mathcal{X}, \mathbf{r}, \mathbf{a})|_c$.*

We define a subscheme $\tilde{\mathcal{P}} \subset \mathcal{H}$:

$$\tilde{\mathcal{P}} = \{(X, \iota, \mathbf{x}) \in \mathcal{H} \mid (X, \mathbf{x}, \mathbf{a}) \text{ weighted pointed stable curves, } \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k} \cong \iota^* \mathcal{O}_{\mathbb{P}^m}(1)\}$$

A direct check shows that $\tilde{\mathcal{P}}$ is a smooth, locally closed, and G -invariant subscheme of \mathcal{H} . We let $\mathcal{P} \subset \tilde{\mathcal{P}}$ be the open subset of (X, ι, \mathbf{x}) such that X are smooth. By definition, we have $\mathcal{P} = \mathcal{K}$.

Lemma 6.15. *The composition $F : \mathcal{P} \rightarrow \overline{\mathcal{K}} \rightarrow \overline{\mathcal{K}}//G$ extends to a unique morphism $\tilde{F} : \tilde{\mathcal{P}} \rightarrow \overline{\mathcal{K}}//G$.*

Proof. Applying deformation theory of nodal curves, we know that \mathcal{P} is dense in $\tilde{\mathcal{P}}$. Let $\Gamma \subset \mathcal{P} \times \overline{\mathcal{K}}//G$ be the graph of the morphism F stated in the Lemma; we let

$$\overline{\Gamma} \subset \tilde{\mathcal{P}} \times \overline{\mathcal{K}}//G$$

be the closure of Γ . Let $p : \overline{\Gamma} \rightarrow \tilde{\mathcal{P}}$ be the projection. We claim that p is bijective. Indeed, given $\xi = (X, \iota, \mathbf{x}) \in \tilde{\mathcal{P}}$, we let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1), \mathbf{r})$ be the family given by Proposition 6.5, which shows that $\xi \in p(\overline{\Gamma})$. This proves that p is surjective. On the other hand, repeating the proof of Lemma 6.13, we see that p is one-to-one. This proves that p is bijective.

Next, we claim that p is an isomorphism. Since $\tilde{\mathcal{P}}$ is smooth, $\mathcal{P} \subset \tilde{\mathcal{P}}$ is dense, and Γ is isomorphic to \mathcal{P} , we conclude that $\overline{\Gamma}$ is reduced. Then since $p : \overline{\Gamma} \rightarrow \tilde{\mathcal{P}}$ is birational, a diffeomorphism and $\tilde{\mathcal{P}}$ is smooth, p must be étale. Thus p is an isomorphism. Finally, by composing the isomorphism p^{-1} with the projection to the second factor of $\tilde{\mathcal{P}} \rightarrow \overline{\mathcal{K}}//G$, we obtain the desired extension \tilde{F} of F . \square

Proof of Proposition 6.14. We cover T by a collection of affine open $\{T_a\}_{a \in A}$. Let $\pi_a : \mathcal{X}_a \rightarrow T_a$ with sections $\mathbf{r}_{a,i} : T_a \rightarrow \mathcal{X}_a$ be the restriction of \mathbf{r}_i to T_a of the family on T . By fixing a trivialization $\pi_{a*} \omega_{\mathcal{X}_a/T_a}(\mathbf{a} \cdot \mathbf{r}_a)^{\otimes k} \cong \mathcal{O}_{T_a}^{\oplus m+1}$, we obtain morphisms $f_a : T_a \rightarrow \tilde{\mathcal{P}}$. Composed with the morphism \tilde{F} constructed in the previous Lemma, we obtain $\tilde{F} \circ f_a : T_a \rightarrow \overline{\mathcal{K}}//G$.

Since the choice of picking the trivializations does not alter the morphism $\tilde{F} \circ f_a$, this collection $\{\tilde{F} \circ f_a\}_{a \in A}$ patches to a morphism $T \rightarrow \overline{\mathcal{K}}//G$. This proves the first part of Proposition 6.14.

Finally, that $\psi(f(c))$ is the point associated to the weighted pointed curve $(\mathcal{X}, \mathbf{r}, \mathbf{a})|_c$ follows from the construction. \square

Proof of Theorem 6.4. It follows from Proposition 6.5, 6.14, and Lemma 6.13. \square

For completeness, we describe without proof the geometry of poly-stable³ points in \mathcal{C}^{ss} .

³Recall a point $\xi \in \mathcal{C}^{ss}$ is poly-stable if the G -orbit $G \cdot \xi$ is closed in \mathcal{C}^{ss} .

Definition 6.16 ([2] when $\mathbf{x} = \emptyset$). *We call $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ extremal if for any proper subcurve $Y \subset X$ satisfying $\vec{\delta}_Y(\mathcal{O}_X(1)) = \mathbf{M}_Y^-$ (cf. (6.11)), we have $L_Y = Y \cap Y^\complement \subset E_X$, where E_X is the union of degree one rational curves in $X \subset \mathbb{P}W$.*

Let $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ be a slope semi-stable weighted pointed nodal curve such that $\deg X > M$ with M given in Theorem 1.5. Then the Chow point of $(X, \mathcal{O}_X(1), \mathbf{x})$ is poly-stable with respect to the polarization $\mathcal{O}_\Xi(1, \mathbf{a})$ if and only if it is extremal. (This in case $\mathbf{x} = \emptyset$ was proved by Caporaso in [2].)

7. K -STABILITY OF NODAL CURVE

In this section, we give another application of Theorem 1.5, which was motivated by a question of Yuji Odaka on studying the K -stability of a polarized nodal curve.

Theorem 7.1. *A polarized connected nodal curve $(X, \mathcal{O}_X(1))$ is K -stable if and only if $\mathcal{O}_X(1)$ is numerically equivalent to a multiple of ω_X .*

We comment that Odaka has proved the K -stability for nodal curve X polarized by $\mathcal{O}_X(1) = \omega_X^{\otimes k}$ for some $k \in \mathbb{N}$ [18]. His proof uses birational geometry and a weight formula proved by himself and by the second named author independently. He also informed us that he was able to generalize his method to prove the above theorem.

We recall the notion of K -stability of polarized varieties.

Definition 7.2 ([19, Sect. 3]). *A test configuration for a polarized scheme $(X, \mathcal{O}_X(1))$ consists of a \mathbb{C}^* -equivariant flat projective morphism $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$, where \mathbb{C}^* acts on \mathbb{A}^1 via the usual action, and a \mathbb{C}^* -linearized relative very ample line bundle \mathcal{L} of π , such that for any $t \neq 0 \in \mathbb{A}^1$, $(\mathcal{X}_t, \mathcal{L}_t) \cong (X, \mathcal{O}_X(1))$. (Here $\mathcal{L}_t = \mathcal{L}|_{\mathcal{X}_t}$.) We call such test configuration $(\mathcal{X}, \mathcal{L})$ a product test configuration if $\mathcal{X} \cong X \times \mathbb{A}^1$; we call it a trivial test configuration if in addition to that it is a product test configuration, the line bundle \mathcal{L} is a pull back from X and the \mathbb{C}^* -action is the product action that acts trivially on X .*

For notational simplicity, from now on we restrict ourselves to when $(X, \mathcal{O}_X(1))$ is a polarized nodal curve. Given a test configuration $(\mathcal{X}, \mathcal{L})$, we let $w(l)$ be the weight of the induced \mathbb{C}^* -action on $\pi_* \mathcal{L}^{\otimes l}|_0$; $w(l) = a_2 l^2 + a_1 l + a_0$ is a degree 2 ($= \dim X + 1$) polynomial in l . We then form the quotient

$$\frac{w(l)}{l \cdot \chi(\mathcal{O}_X(l))} = e_0 + e_{-1} l^{-1} + \dots$$

Definition 7.3. *We define the Donaldson-Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X(1))$ be*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = e_{-1} = -\frac{a_{n+1}b_{n-1} - a_n \cdot b_n}{b_n^2};$$

the polarized nodal curve $(X, \mathcal{O}_X(1))$ is K -stable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X(1))$.

For $(X, \mathcal{O}_X(1))$, and letting $W^\vee = H^0(\mathcal{O}_X(l))$ with $X \subset \mathbb{P}W$ the tautological embedding, then given any 1-PS subgroup λ of $\mathrm{Aut} \mathbb{P}W$, the \mathbb{C}^* -orbit of X in $\mathbb{P}W \times \mathbb{A}^1$ via the diagonal \mathbb{C}^* action produces a test configuration of $(X, \mathcal{O}_X(1))$; we denote such test configuration by $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$.

Conversely, given $(X, \mathcal{O}_X(1))$, any test configuration of $(X, \mathcal{O}_X(1))$ can be derived from a 1-PS of $\mathrm{Aut} \mathbb{P}W$ (cf. [19, Prop. 3.7]). Thus to prove the K -stability of $(X, \mathcal{O}_X(1))$, it

suffices to show that the Donaldson-Futaki invariant $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < 0$ for all 1-PS λ of Aut PW .

Our starting point is to relate $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ with the Chow weights of $(X, \mathcal{O}_X(l))$. We pick a λ -diagonalizing basis $\mathbf{s} = \{s_0, \dots, s_m\}$ of W^\vee ; namely, under its dual bases the action λ is given by the following (1-PS of $GL(W^\vee)$):

$$(7.1) \quad \lambda(t) := \text{diag}[t^{\rho_0}, \dots, t^{\rho_m}], \quad \rho_0 \geq \rho_1 \geq \dots \geq \rho_m = 0, \quad \rho_i \in \mathbb{Z}.$$

Here by shifting all ρ_i by a common integer a , we can choose $\rho_m = 0$.

Our next step is to construct a 1-PS of $W_l^\vee = H^0(\mathcal{O}_X(l))$ that is most close to λ . First, since $(X, \mathcal{O}_X(1))$ is K -stable is equivalent to that of $(X, \mathcal{O}_X(k))$, $k > 0$. Thus without lose of generality, we can assume $\mathcal{O}_X(1)$ is sufficiently ample so that

$$(7.2) \quad \phi_l : S^l W^\vee \longrightarrow W_l^\vee = H^0(\mathcal{O}_X(l)).$$

For convenience, for multi-indices $I = (i_0, \dots, i_m)$, we denote $s^I = s_1^{i_0} \dots s_m^{i_m}$; then s^I has total degree $|I| = \sum i_j$. For the weights $\rho = (\rho_0, \dots, \rho_m)$, we denote $\rho(I) = \sum \rho_j \cdot i_j$, which is the weight of s^I under the induced λ action on $S^l W^\vee$.

We let \mathfrak{S}_l be the set of monomials in $S^l W^\vee$. We order \mathfrak{S}_l as follows: $s^I \succ s^{I'}$ when either $\rho(I) < \rho(I')$, or when $\rho(I) = \rho(I')$ and there is a $0 \leq j_0 \leq m$ such that $i_j = i'_j$ for all $j > j_0$ and $i_{j_0} > i'_{j_0}$. Thus, s_m^l (resp. s_0^l) is the largest (resp. least) element in \mathfrak{S}_l . Further, $s^I \succ s^{I'}$ if and only if $s^J \cdot s^I \succ s^J \cdot s^{I'}$ for any non-trivial monomial s^J , and vice versa.

We pick a basis of W_l^\vee , which will be a diagonalizing basis of the λ_l we will construct momentarily. Let $m_l + 1 = \dim W_l^\vee$. We set $s_{l, m_l} = s_m^l$, with weight $\varrho_{l, m_l} = \rho_m^l$. Suppose for an integer $0 \leq k < m_l$, we have picked $s_{l, k+1}, \dots, s_{l, m_l}$ and their weights $\varrho_{l, j}$, we let $s_{l, k} = s^{I_k}$ be the largest element in

$$\{s^I \in \mathfrak{S}_l \mid \phi_l(s^I) \notin \phi_l(\Theta_{l, k+1})\},$$

where $\Theta_{l, k+1} = \phi_l(\mathbb{C}\{s_{l, k+1}, \dots, s_{l, m_l}\})$, and $\mathbb{C}\{\cdot\}$ is the \mathbb{C} -linear span of elements in $\{\cdot\}$. We let $\rho(I_k)$, which is the weight of s^{I_k} in $S^l W^\vee$ under $\lambda^{\otimes l}$.

We let $s_{l, k} = \phi_l(s^{I_k})$. Then $s_{l, 0}, \dots, s_{l, m_l}$ form a basis W_l^\vee . By setting

$$(7.3) \quad \lambda_l(\sigma) \cdot s_{l, k} = \sigma^{\varrho_{l, k}} s_{l, k}$$

we obtain a 1-PS of Aut PW_l of diagonalizing basis $\{s_{l, 0}, \dots, s_{l, m_l}\}$.

Following the discussion in Section 2, we define, (for $q \in \tilde{X}$ and $\tilde{s}_{l, k}$ the lift of $s_{l, k}$ to the normalization \tilde{X} of X),

$$(7.4) \quad \tilde{h}_\alpha(\lambda_l) = \min\{i \mid \tilde{s}_{l, i+1}|_{\tilde{X}_\alpha} = 0\}, \quad \tilde{h}(\lambda_l, q) = \max\{i \mid v(\tilde{s}_{l, i}, q) \neq \infty\},$$

and

$$(7.5) \quad \tilde{\Lambda}_\alpha(\lambda_l) = \{p \in \tilde{X}_\alpha \mid \tilde{s}_{l, \tilde{h}_\alpha(\lambda_l)}(p) = 0\}, \quad \tilde{\Lambda}(\lambda_l) = \cup_{\alpha=1}^r \tilde{\Lambda}_\alpha(\lambda_l).$$

(Here r is the number of irreducible components of X .)

Lemma 7.4. *For the 1-PS λ_l constructed, we have $\tilde{\Lambda}(\lambda_l) = \tilde{\Lambda}(\lambda)$. Further, for each $q \in \tilde{\Lambda}(\lambda)$, and for $w(\tilde{\mathcal{J}}, q)$ defined in (2.9), we have $v(\tilde{s}_{l, \tilde{h}(\lambda_l, q)}, q) = w(\tilde{\mathcal{J}}^l, q) = l \cdot w(\tilde{\mathcal{J}}, q) = l \cdot v(\tilde{s}_{\tilde{h}(q)}, q)$, and hence $e(\mathcal{J}(\lambda_l)) = \text{n.l.c. } \chi(\mathcal{O}_{X \times \mathbb{A}^1}(k)/\mathcal{J}(\lambda)^{kl})$.*

Let λ'_l be the staircase 1-PS obtained from λ_l by applying Proposition 3.6, then

- (1) the support $\tilde{\Lambda}(\lambda_l)$ is the same as $\tilde{\Lambda}(\lambda'_l)$; for each $q \in \tilde{\Lambda}(\lambda_l)$, $w(\tilde{\mathcal{J}}(\lambda_l), q) = w(\tilde{\mathcal{J}}(\lambda'_l), q)$;
- (2) for each $q \in \tilde{\Lambda}(\lambda_l)$, $\Delta_q(\lambda_l) = l \cdot \Delta_q(\lambda) \subset \Delta_q(\lambda'_l)$.

Proof. Because of our choice of λ_l , we have the middle identity (the first the the third is by the definition)

$$(s_0, \dots, s_m) = \mathcal{J}(\lambda_l) = \mathcal{J}(\lambda)^l = (t^{\rho_0} s_0, \dots, t^{\rho_m} s_m)^l \subset \mathcal{O}_{X \times \mathbb{A}^1}(l).$$

This prove the first part of the Lemma. The second part, follows from the construction of staircase in Proposition 3.6, and from Lemma 2.5. \square

We have the following useful Lemma, relating $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ to the weights $\omega(l)$ of λ .

Lemma 7.5 ([5, Sect. 2.3], [19, Thm. 3.9]). *Let $X \subset \mathbb{P}W^\vee$ and λ a 1-PS be as before. Then*

$$\lim_{l \rightarrow \infty} l^{-1} \cdot \omega(\lambda_l) = -b_1^{-1} \cdot \text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < \infty.$$

Thus to prove $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < 0$, it suffices to show that

$$(7.6) \quad \lim_{l \rightarrow \infty} l^{-1} \cdot \omega(\lambda_l) > 0.$$

Let λ'_l be the staircase constructed from λ_l using Proposition 3.6, of the same weights $\varrho_{l,i}$. We let $\hat{\varrho}_{l,i}$ be the shifted weights according to the rule (5.9) applied to λ'_l ; namely, $\hat{\varrho}_{l,i} = \min_\alpha \{\varrho_i - \varrho_{\hbar_\beta(\lambda'_l)} \mid i \in \mathbb{I}_\beta(\lambda'_l)\}$.

Proof of Theorem 7.1. Suppose X is a stable (nodal) curve and $\mathcal{O}_X(1)$ is numerically proportional to ω_X , then $(X, \mathcal{O}_X(1))$ is slope stable. We will show in this case that for any 1-PS $\lambda \in SL(W)$ we have $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < 0$.

The first case to study is when $e(\mathcal{J}(\lambda)) = 0$. By Section 2 we know that there is a $0 < i_0 < m$ such that $\varrho_{i_0} = 0$ and $\bigcap_{k \geq i_0} \{s_k = 0\} = \emptyset$. Stoppa proved that in this case either the test configuration $(\mathcal{X}_\lambda, \mathcal{L})$ induced by λ is trivial (cf. Definition 7.2) or $\text{DF}(\lambda) < 0$ [21, page 1405-1406]. This settles this case.

The other case is when $e(\mathcal{J}(\lambda)) > 0$. Applying Theorem 5.8 (since $(X, \mathcal{O}_X(1))$ is slope stable), and applying Proposition 3.6, we can find an $\epsilon > 0$ so that

$$(7.7) \quad l^{-1} \cdot \omega(\lambda_l) \geq l^{-1} \cdot \omega(\lambda'_l) \geq \frac{1}{\deg X + l^{-1}(1 - g_X)} \cdot \frac{\epsilon}{l^2} \cdot \sum_{i=0}^{m_l} \hat{\varrho}_{l,i}.$$

Thus $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < 0$ follows from Lemma 7.6, which we will prove shortly. This proves the *if* part, once we prove Lemma 7.6.

For the other direction, suppose $(X, \mathcal{O}_X(1))$ is K -stable, we show that $(X, \mathcal{O}_X(1))$ is slope stable. Suppose $(X, \mathcal{O}_X(1))$ is not slope stable, then there is a subcurve $Y \subset X$ destabilizing the polarized curve $(X, \mathcal{O}_X(1))$, namely,

$$(7.8) \quad \frac{\deg_Y \omega_X}{\deg \omega_X} \cdot \deg X - \deg Y - \frac{\ell_Y}{2} \geq 0.$$

Let $H^0(\mathcal{O}_X(1)|_Y)^\vee \subset W = H^0(\mathcal{O}_X(1))^\vee$, which is the linear subspace spanned by Y ; let $m_0 + 1 = \dim H^0(\mathcal{O}_X(1)|_Y)$. We choose a two-weight 1-PS λ as in the proof of Theorem 1.5 (at the end of Section 5) so that λ acts of weight 1 on $H^0(\mathcal{O}_X(1)|_Y)^\vee \subset H^0(\mathcal{O}_X(1))^\vee$ and acts of weight 0 on its linear complement; we form its associated test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$. We now evaluate

$$\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) = \lim_{l \rightarrow \infty} -\frac{\omega(\lambda_l)}{l} = \lim_{l \rightarrow \infty} \frac{1}{l} \cdot \left(\frac{2l \deg X \sum_{i=0}^{m_l} \varrho_{l,i}}{l \deg X + 1 - g} - e(\mathcal{J}(\lambda_l)) \right).$$

First, the central fiber \mathcal{X}_0 is the union $Y \cup E \cup Y^{\mathbb{C}}$, where E consists of ℓ_Y number of lines that inserted in the linking nodes L_Y of X ; the total space $H^0(\mathcal{L}_\lambda^{\otimes l}|_{\mathcal{X}_0})$ has a decomposition

$$H^0(\mathcal{L}_\lambda^{\otimes l}|_{\mathcal{X}_0}) \cong H^0(\mathcal{O}_X(l)|_Y) \oplus H^0(\mathcal{O}_E(l)(-Y \cap E)) \oplus H^0(\mathcal{O}_X(l)|_{Y^{\mathbb{C}}}(-E \cap Y^{\mathbb{C}}));$$

elements in $H^0(\mathcal{O}_X(l)|_Y)$ have weights l ; elements in $H^0(\mathcal{O}_E(l)(-Y \cap E))$ have weights $l-1, \dots, 0$ for each copy in E ; elements in $H^0(\mathcal{O}_X(l)|_{Y^{\mathbb{C}}}(-E \cap Y^{\mathbb{C}}))$ have weights 0. Thus

$$\sum \varrho_{l,i} = h^0(\mathcal{O}_X(l)|_Y) \cdot l + \ell_Y \cdot \frac{l(l-1)}{2} = (\deg Y + \frac{\ell_Y}{2}) \cdot l^2 + (1 - g_Y - \frac{\ell_Y}{2}) \cdot l.$$

Using that $\sum_{i=0}^{m_l} \varrho_{l,i} = e(\mathcal{J}(\lambda)) \cdot \frac{l^2}{2} + \text{lower order term}$, and simplifying, we obtain

$$(7.9) \quad \text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) = \frac{g-1}{\deg X} \left(\frac{\deg_Y \omega_X}{\deg \omega_X} \cdot \deg X - \deg Y - \frac{\ell_Y}{2} \right).$$

Since $Y \subset X$ is destabilizing, by (7.8) we have $\text{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) > 0$, violating that $(X, \mathcal{O}_X(1))$ is K -stable. This proves that $(X, \mathcal{O}_X(1))$ is slope stable.

On the other hand, since K -stability of $(X, \mathcal{O}_X(l))$ is independent of $l > 0$, $(X, \mathcal{O}_X(1))$ is K -stable implies that $(X, \mathcal{O}_X(l))$ is K -stable for $l > 0$, thus $(X, \mathcal{O}_X(l))$ is slope stable.

Once we know that $(X, \mathcal{O}_X(l))$ is slope stable for large l , an easy argument shows that $\mathcal{O}_X(l)$ satisfies (1.6) for all large l , which is possible only when $\mathcal{O}_X(1)$ is numerically proportional to ω_X . This proves the Theorem. \square

Lemma 7.6. *Let the notation be as stated. Suppose further that $e(\mathcal{J}(\lambda)) > 0$. Then⁴*

$$\lim_{l=l_k \rightarrow \infty} \frac{1}{l^2} \cdot \sum_{i=0}^{m_l} \hat{\varrho}_{l,i} > 0.$$

Proof. We comment that in case X is irreducible, the positivity is immediate. Indeed, applying [16, Prop. 2.11], we have

$$(7.10) \quad \sum_{i=0}^{m_l} \varrho_{l,i} = e(\mathcal{J}(\lambda)) \cdot \frac{l^2}{2} + a_1 \cdot l + a_2, \quad a_i \text{ depending only on } \lambda.$$

Suppose X is irreducible, we have $\varrho_{l,i} = \hat{\varrho}_{l,i}$. Therefore,

$$\lim_{l=l_k \rightarrow \infty} \frac{1}{l^2} \cdot \sum_{i=0}^{m_l} \hat{\varrho}_{l,i} = \lim_{l=l_k \rightarrow \infty} \frac{1}{l^2} \cdot \sum_{i=0}^{m_l} \varrho_{l,i} = \frac{e(\mathcal{J}(\lambda))}{2} > 0.$$

We now prove the general case. We claim that there is a $1 \leq \beta \leq r$ and $q \in \tilde{X}_\beta$ so that

$$(7.11) \quad |\Delta_q(\lambda)| - \rho_{\tilde{h}_\beta(\lambda)} \cdot w(\tilde{\mathcal{J}}(\lambda), q) > 0.$$

Suppose for any $q \in \tilde{X}_\beta$ the inequality (7.11) does not hold. Since the \geq always hold, we will have that $\varrho_i = \varrho_{\tilde{h}_\alpha}$ for every $i \in \mathbb{I}_\alpha$. Since $e(\mathcal{J}(\lambda)) > 0$, we must have an $\alpha > 1$ such that $\varrho_{\tilde{h}_\alpha} > 0$. Since X is connected, we can find a pair $\alpha \neq \beta$ so that $X_\alpha \cap X_\beta \neq \emptyset$, and $\rho_{\tilde{h}_\alpha(\lambda)} > \rho_{\tilde{h}_\beta(\lambda)} = 0$.

We next let $q \in \tilde{X}_\beta$ be a lift of a node in $X_\alpha \cap X_\beta$. We show that the pair (β, q) satisfies the inequality (7.11). Let $\pi : \tilde{X} \rightarrow X$ be the projection. Since $\pi(q) \in X_\alpha$, we have $\tilde{s}_j(q) = 0$ for all $j > \tilde{h}_\alpha(\lambda)$; since $\rho_{\tilde{h}_\alpha(\lambda)} > \rho_{\tilde{h}_\beta(\lambda)} = 0$, we have $i_0(q) \leq \tilde{h}_\alpha(\lambda)$. Thus

⁴Here by $l = l_k \rightarrow \infty$ we mean that by passing to a subsequence we assume that the limit $\lim_{l_k \rightarrow \infty}$ does exist, and is finite.

$\rho_{i_0(q)} \geq \rho_{h_\alpha(\lambda)} > 0$, and then $\Delta_q(\lambda)$ is two dimensional. Since $\rho_{h_\beta(\lambda)} = 0$, this contradicts to initial assumption that (7.11) never holds. This proves the claim.

Let (β, q) be a pair satisfying (7.11). We next establish the following two inequalities

$$(7.12) \quad \sum_i \hat{\varrho}_{l,i} \geq \frac{1}{2} (|\Delta_q(\lambda'_l)| - \varrho_{l, h_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q))$$

and

$$(7.13) \quad \lim_{l=l_k \rightarrow \infty} l^{-2} \cdot (|\Delta_q(\lambda'_l)| - \varrho_{l, h_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q)) \geq |\Delta_q(\lambda)| - \varrho_{l, h_\beta(\lambda)} \cdot w(\tilde{\mathcal{J}}(\lambda), q).$$

We prove the inequality (7.12). Following the notation introduced in Section 4, we have

$$\sum_{i=0}^{m_l} \hat{\varrho}_{l,i} \geq \sum_{i \in \mathbb{I}_\alpha(\lambda'_l)} \hat{\varrho}_{l,i} \geq \sum_{i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)} \hat{\varrho}_{l,i},$$

where $\mathbb{I}_\beta(\lambda'_l)$ is the set of indices for \tilde{X}_β , and $\mathbb{I}_q^{\text{pri}}(\lambda'_l)$ is the set of primary indices for $q \in \tilde{X}_\beta$, both with respect to the staircase λ'_l .

By Proposition 3.11 and 3.13, we know that for $i_0(q) \neq i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)$, we have $\hat{\varrho}_{l,i} = \varrho_{l,i} - \varrho_{l, h_\beta(\lambda'_l)}$. (Note $\hat{\varrho}_{l,i_0(q)} = \varrho_{l,i_0(q)} - \varrho_{l, h_{\alpha'}}(\lambda'_l)$ possibly for some $\alpha' \neq \beta$.)

By the proof of Lemma 4.2, we have

$$\sum_{i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)} \hat{\varrho}_{l,i} - \hat{\varrho}_{l,i_0(p)} \geq |\Delta_q^{\text{pri}}(\lambda'_l) \cap ([1, w^{\text{pri}}(p, \lambda'_l)] \times \mathbb{R})| - \varrho_{l, h_\beta(\lambda'_l)} \cdot (w^{\text{pri}}(p, \lambda'_l) - 1).$$

Following (4.9), we continue to denote $\bar{j}_p(\lambda'_l) = \max\{i \in \mathbb{I}_p^{\text{pri}}(\lambda'_l)\}$ and $w^{\text{pri}}(p, \lambda'_l) = w(\tilde{\mathcal{E}}_{\bar{j}_p(\lambda'_l)+1}(\lambda'_l), p)$. By the boundness result from Corollary 3.14, for sufficiently large l , the effects to the shape of $\Delta_q(\lambda'_l)$ from the *secondary indices* $\mathbb{I}_q(\lambda'_l) \setminus \mathbb{I}_q^{\text{pri}}(\lambda'_l)$ is marginal, thus for large l we have

$$|\Delta_q^{\text{pri}}(\lambda'_l) \cap ([1, w^{\text{pri}}(q, \lambda'_l)] \times \mathbb{R})| - \varrho_{l, h_\beta(\lambda'_l)} \cdot (w^{\text{pri}}(q, \lambda'_l) - 1) \geq \frac{1}{2} (|\Delta_q(\lambda'_l)| - \varrho_{l, h_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q)).$$

Combined, and adding $\hat{\varrho}_{l,i_0(p)} > 0$, we obtain

$$\sum_{i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)} \hat{\varrho}_{l,i} \geq \frac{1}{2} (|\Delta_q(\lambda'_l)| - \varrho_{l, h_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q)).$$

This proves (7.12).

Before we move to (7.13), we claim that

$$(7.14) \quad A_\beta := \lim_{l=l_k \rightarrow \infty} \frac{\varrho_{l, h_\beta(\lambda'_l)} - \varrho_{l, h_\beta(\lambda_l)}}{l} = 0.$$

Suppose not, say $A_\beta > 0$, (it is non-negative,) then for $l = l_k$ large,

$$\varrho_{l, h_\beta(\lambda'_l)} - \varrho_{l, h_\beta(\lambda_l)} \geq \frac{1}{2} \cdot l \cdot A_\beta;$$

by examining the geometry of $\Delta_q(\lambda_l) \subset \Delta_q(\lambda'_l)$, we obtain

$$|\Delta_q(\lambda'_l)| - |\Delta_q(\lambda_l)| \geq \frac{A_\beta}{2} \cdot \frac{l \cdot w(\tilde{\mathcal{J}}(\lambda), q)}{h_{\Delta_q(\lambda)} - \varrho_{l, h_\beta(\lambda)}} \cdot l := C \cdot l^2 > 0,$$

where $h_{\Delta_q(\lambda)}$ is the height of $\Delta_q(\lambda)$. This implies

$$l^{-1} \cdot \omega(\lambda_l) = l^{-1} \cdot \omega(\lambda'_l) + l^{-1} \cdot (e(\mathcal{J}(\lambda'_l)) - e(\mathcal{J}(\lambda_l))) \geq l^{-1} \cdot (e(\mathcal{J}(\lambda'_l)) - e(\mathcal{J}(\lambda_l))),$$

where we have used Theorem 1.5 to deduce $\omega(\lambda'_l) \geq 0$.

By Corollary 2.7 and our construction of staircase using Proposition 3.6, we deduce

$$l^{-1} \cdot (e(\mathcal{J}(\lambda'_l)) - e(\mathcal{J}(\lambda_l))) \geq l^{-1} \cdot (|\Delta_q(\lambda'_l)| - |\Delta_q(\lambda_l)|) > C \cdot l.$$

This is impossible since Lemma 7.5 implies that the left-hand-side remains bounded as $l = l_k \rightarrow \infty$. So we must have $A_\beta = 0$. This proves the claim.

We prove inequality (7.13). Because $A_\beta = 0$, $|\Delta_q(\lambda'_l)| \geq |\Delta_q(\lambda_l)|$, and by Lemma 7.4, we obtain

$$\begin{aligned} & |\Delta_q(\lambda'_l)| - \varrho_{l, \hbar_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q) \\ &= |\Delta_q(\lambda'_l)| - \varrho_{l, \hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathcal{J}}(\lambda_l), q) + \varrho_{l, \hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathcal{J}}(\lambda_l), q) - \varrho_{l, \hbar_\beta(\lambda'_l)} \cdot w(\tilde{\mathcal{J}}(\lambda'_l), q) \\ &\geq |\Delta_q(\lambda_l)| - \varrho_{l, \hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathcal{J}}(\lambda_l), q) + (\varrho_{l, \hbar_\beta(\lambda_l)} - \varrho_{l, \hbar_\beta(\lambda'_l)}) \cdot w(\tilde{\mathcal{J}}(\lambda_l), q) \\ &= l^2 \cdot (|\Delta_q(\lambda)| - \varrho_{l, \hbar_\beta(\lambda)} \cdot w(\tilde{\mathcal{J}}(\lambda), q)) + l^2 \cdot \frac{\varrho_{l, \hbar_\beta(\lambda_l)} - \varrho_{l, \hbar_\beta(\lambda'_l)}}{l} \cdot w(\tilde{\mathcal{J}}(\lambda), q). \end{aligned}$$

Taking limit as $l = l_k \rightarrow \infty$, and using $A_\beta = 0$, we obtain (7.13).

Finally, by (7.10), and that $0 \leq \hat{\varrho}_{l,i} \leq \varrho_{l,i}$, we conclude that the limit in the Lemma is finite; thus the limit is finite and positive; this proves the Lemma. \square

Remark 7.7. *Following [16, Sect. 3], (or the recent work of Odaka [18],) we know that a K -stable polarized curve has at worst nodal singularity. Thus results like Theorem 7.1 show that K -stability compactifies the moduli of smooth curves (of $g \geq 2$). As K -stability is an analytic version of GIT stable via a CM-line bundle defined by Paul and Tian [20], which is a multiple of $\lambda^{\otimes 12} \otimes \delta^{-1}$ for moduli of curves (cf. [16, Thm. 5.10]), Theorem 7.1 can be viewed as comparing compactifications via two versions of GIT stability, one via finite dimensional embedding and one via analysis. Generalizing this to high dimensional canonically polarized varieties remains a challenge. Lately, Yuji Odaka (cf. [18]) has made important progress along this direction.*

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